SIER Working Paper Series No. 113

Learning Rival's Information in Interdependent Value Auctions

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September, 2017

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September 13, 2017

Abstract

We study a simple auction model with interdependent values in which bidders can learn their rival's information and compete in the first-price or second-price auction. We characterize unique symmetric equilibrium strategies—both learning and bidding strategies—for the two auction formats. While bidders learn rival's signals with higher probabilities in the first-price auction, they earn higher rent in the second-price auction. We also show that when learning cost is small, signal correlation is low, or value interdependence is weak, the first-price auction generates a higher revenue than the second-price auction, while the revenue ranking is reversed otherwise.

1 Introduction

In August 2013, the "big three" mobile network operators in Korea—SK Telecom, KT, and LG Uplus—competed for long-term evolution (LTE) wireless spectrum bands in a spectrum auction in Korea. Since KT was lagging behind its competitors in the LTE market at that moment, it was imperative for the company to get an extra spectrum block that could be

*We wish to thank Yeon-Koo Che, Chang-Koo Chi, Dongkyu Chang, Elma Wolfstetter, Jingfeng Lu, and Sergio Parreiras for their comments. We have also benefited from comments made by seminar participants at City University of Hong Kong, HKUST, Kobe University, Korea University, Summer Meeting of the Korean Econometric Society, and Asian Meeting of the Econometric Society. Koh acknowledges the support for this work by the Ministry of Education of the Republic of Korea and the National Research Foundation of Korea (NRF-2017S1A5A8020721).

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combined with its existing block to provide the LTE service. The first thing the company did to prepare a bid in the auction was to form a task force that worked intensively to estimate the values of the spectrum blocks to its rivals as well as the company itself. Also, the task force advised the bid team throughout the bidding process and, to do so, they paid close attention to what the true values of the rival companies could be.[1] In the end, KT won the desired block by outbidding a rival company by less than a couple of million dollars in the contest where the three companies ended up paying more than two billion dollars in total.

We believe that the above story illustrates just one of many instances where bidders try to acquire information about rivals. While the information acquisition in auctions has been an important issue in the literature, there are few studies that investigate the acquisition of information about rival bidders.[2] Learning rival's information is important in two aspects: first, the learners can estimate the value of an auctioned object more precisely, gaining an *informational advantage*; second, they can better predict the bidding strategy of their rival, gaining a *strategic advantage*. Notice that the first aspect becomes important to the extent that one's value depends on his rival's information, that is, the values are *interdependent*. In the current paper, we study how the two aspects of learning work together to affect the bidders' incentive to learn their rival's information in standard auctions—first-price and second-price auctions—with interdependent values and thereby affect the performance of the two auction formats.

To this end, we consider a simple model in which there are two bidders, who are ex ante symmetric, competing for a single object. Each bidder is informed of a binary signal which is correlated with the other's signal. The value of the object for each bidder is given as a linear combination of his own signal and his rival's signal with more weight assigned to the former.[3] The weight assigned to the rival's signal measures the degree of value interdependence, capturing the private and common values as two polar cases. Our model has a simple time line: Initially, for a given auction format, each bidder decides whether to learn the other bidder's signal by incurring a cost. This decision is unobservable, i.e., the information acquisition is *covert*. Next, bidders simultaneously decide how much to bid based on their information. Lastly, the winner is announced and trade occurs according to the auction format. This is the main setup for our study, denoted as I^2. In the paper, we

[1] The auction format was a variant of the simultaneous ascending auction, followed by a one-shot, seal-bid stage.

[2] We will later review the literature on information acquisition in auctions and mechanism design in general.

[3] This implies that whoever holds a higher signal has a higher value for the object.

also consider an alternative setup, denoted as I^1, in which bidders are informed of no prior signal and can incur a cost to learn their own signal. This setup has been studied in the auction literature (for instance, Persico (2000) and Shi (2012) among others) and will be used as a benchmark to compare the results from the main setup.

Compared to the benchmark case, our main setup, I^2, has a couple of crucial differences: Since bidders are informed of prior signals, they will have multi-dimensional information in the bidding stage after learning their rival's signal.[4] Moreover, the learning decision of each bidder is dependent on his prior signal, which makes the equilibrium characterization and analysis highly nontrivial. Although the simplicity of our setup leaves a question of generalization, it is instrumental for obtaining clear intuition about how the possibility of acquiring rival's signal affects bidders' learning and bidding behavior through the two channels—informational and strategic advantages. Also, it enables us to conduct various comparative statics analyses.

In the analysis of our main setup, we characterize a unique (mixed-strategy) symmetric equilibrium, consisting of learning and bidding strategies, for the two auction formats. To explain, let us call a bidder with higher (resp., lower) prior signal *strong* (resp., *weak*) bidder. For the second-price auction, bidders never learn their rivals' signal and follow the same bidding strategy as in the setup without the learning possibility, which results in an efficient allocation, i.e., a strong bidder always wins against a weak bidder. Some interesting features emerge from the equilibrium characterization for the first-price auction. First, despite binary signals, the number of bidder types in the bidding stage can increase substantially and is determined endogenously as a result of learning decision. Second, bidders' learning behavior depends on their prior signals. In particular, a strong bidder learns the rival's signal with higher probability than does a weak bidder. Third, unlike the first-price auction without the learning possibility, a strong bidder may bid less aggressively than a weak bidder, in particular when the former learns that his rival is weak while the latter learns that his rival is strong. The resulting allocation is inefficient, and the total surplus is even lower if the learning cost is accounted for.

The learning and bidding behavior in equilibrium can be explained by the aforementioned two advantages. Clearly, the informational advantage is important in any auction format. In contrast, the strategic advantage is more important in the first-price auction where bid-

[4] Although the two signals together determines each bidder's value (which is single-dimensional), they cannot be reduced to such single-dimensional information, since the rival's signal also conveys the information about his bidding strategy.

ders wish to (optimally) shade their bids and, to do so, need to infer their rival's strategy accurately. Therefore, the equilibrium strategy involves higher learning probabilities in the first-price auction than in the second-price auction.[5] Note also that in the first-price auction, the strategic advantage is more valuable to a strong bidder than to a weak bidder, since the former has a greater room for shading his bid upon learning that his rival is weak. Thus, a strong bidder learns his rival's signal with higher probability than a weak bidder does. Moreover, there is a negative relationship between the two types' learning behaviors: the strong bidder's learning probability tends to increase when the weak bidder's learning probability decreases. This is (partly) because a weak bidder bids less aggressively when not knowing that his rival is strong, which gives his strong rival a greater incentive to learn and shade bid.

Our analysis yields intuitive comparative statics, for which the learning cost and degrees of signal correlation and value interdependence are central parameters. The learning probability decreases in the learning cost, irrespective of bidder types. The weak bidder's learning probability increases as signals become more correlated or values become less interdependent. This can be understood from observing that unlike a strong bidder, a weak bidder derives the benefit of learning mostly from the informational advantage—that is, finding out whether his value is higher than his prior signal would indicate—while learning has greater informational value if the signal correlation is lower or value interdependence is stronger.[6] As mentioned above, when a greater informational advantage induces a weak bidder to learn more and bid more aggressively against his strong rival, it tends to negatively affect the strong bidder's strategic advantage of learning, making him learn less and thereby shade bid less (i.e., bid more aggressively).

For the payoff consequence of the equilibrium learning/bidding behavior, observe first that learning each other's signal reduces bidders' private information, leading to smaller information rent. Thus, bidders earn lower payoff in the first-price auction than in the second-price auction. However, the effect of learning on the seller's revenue is more subtle. We show that the first-price auction generates a higher revenue than the second-price auction when a weak bidder's learning probability is relatively high, which, as explained above, is the case when the learning cost is small, signal correlation is low, and/or value interdependence

[5] Note that the strategic advantage exists even in I^1, since the learning of one's own signal helps predict the rival's (correlated) signal and thereby his bidding strategy. Hence, bidders in the first-price auction learn their own signal with higher probability than in the second-price auction under I^1 as well.

[6] Observe that the lower signal correlation means each bidder's signal is less informative of his rival's signal.

is weak. Otherwise, the second-price auction is revenue-superior. This is consistent with the above observation that in the first-price auction, the weak bidder's learning and its negative effect on the strong bidder's learning induce both types of bidders to bid more aggressively.

Several papers have studied the problem of information acquisition in auctions, but most of them have focused on the problem of learning bidders' own signal, assuming that bidders have no prior information. Stegeman (1996) and Shi (2012) study this problem in the private values setup while Milgrom (1981) and Matthews (1984) do so in the interdependent values setup. Also, Hausch and Li (1993) compare first-price and second-price auctions in a common value setup and show that the seller's revenue is higher in the second-price auction than in the first-price auction. Persico (2000) shows that with affiliated values, the first-price auction provides a stronger incentive for bidders to acquire information than the second-price auction does. Again, these studies assume no private information for bidders prior to their information acquisition.

Closely related to our study, Bergemann and Välimäki (2005) discuss the possibility that bidders engage in costly "espionage" in a private value first-price auction—which refers to the activity of learning other bidder's information. In private values setup, Fang and Morris (2006) and Tian and Xiao (2010) study how the standard auctions are affected when bidders observe their rival's information. Fang and Morris (2006) show that when each of two bidders observes an imperfect signal about the rival's valuation, the first-price auction generates a lower revenue than the second-price auction, which is consistent with our finding that when values are less interdependent, the first-price auction tends to be revenue-inferior.[7] However, unlike the current paper, they assume that the information about rivals is not acquired but exogenously given. Tian and Xiao (2010) extend Fang and Morris (2006) by endogenizing bidders' information acquisition.[8] To the best of our knowledge, our work is the first to study the acquisition of rival's information in interdependent values setup.

The paper is organized as follows. We introduce our model in Section 2. Section 3 analyzes the first-price and the second-price auctions under the setup I^1. Section 4 characterizes equilibrium for the two auction formats under our main setup I^2. The comparison between the two auctions is provided in Section 5. Section 6 concludes the paper. Proofs are provided in the Appendix and Supplementary Material.

[7] Fang and Morris (2006) also present an example in which their revenue ranking is reversed. For this result, however, they assume that the information acquisition is common knowledge among bidders.

[8] Tian and Xiao (2010) compare two specifications: *ex ante* and *interim* information acquisition where bidders can learn their rivals' valuations before and after observing their own valuations, respectively. See also Li and Tian (2008) for an analysis of the second-price auction.

2 The Model

Suppose that there is a single object to be auctioned off to two bidders, 1 and 2. Each bidder $i = 1, 2$ is initially informed of a signal s_i, which takes one of two values, 0 and 1. This signal will sometimes be referred to as bidder i's *prior signal*. We assume that $\text{Prob}(s_i = 0) = \text{Prob}(s_i = 1) = \frac{1}{2}$ for each $i = 1, 2$, and the two signals are correlated as follows: for all $i, j = 1, 2$ with $i \neq j$, and for all $m, m' \in \{0, 1\}$ with $m \neq m'$,

$$\text{Prob}(s_j = m | s_i = m) = 1 - \text{Prob}(s_j = m' | s_i = m) = \alpha \in (\tfrac{1}{2}, 1).$$

Hence, a higher α means a higher correlation between the signals.

The value of the object to each bidder $i = 1, 2$ is given as

$$v_i(s_i, s_j) = \beta s_i + (1 - \beta) s_j, \quad \beta \in [\tfrac{1}{2}, 1],$$

that is, values are *interdependent* in the sense that each bidder's value depends on the other's signal as well as his own (unless $\beta = 1$). Note that when $\beta = \frac{1}{2}$, bidders have a common value, and when $\beta \in (\frac{1}{2}, 1]$, whoever has a higher signal has a higher value for the object. For this reason, a bidder with a low (resp., high) prior signal will be called *weak* (resp., *strong*) bidder. Note that as β decreases, the relative impact of the other's signal on one's value increases, that is, the values become more interdependent. This implies that the knowledge of rival's signal becomes more important for the estimation of one's own value. Note also that as β decreases, the value difference between weak and strong bidders becomes smaller (i.e., values become more common).

We consider two auction formats, first-price and second-price auctions. In both auctions, a bidder who submits a higher bid wins the object, while the winner pays the highest (i.e., his own) bid in the first-price auction and the second-highest (i.e., the rival's) bid in the second-price auction. Ties are broken randomly.

In each auction, our model of information acquisition consists of two stages; the *learning stage* and the subsequent *bidding stage*. In the learning stage, each bidder i decides whether to learn the rival's signal $s_j, j \neq i$, by incurring cost $k > 0$. We assume that whether each bidder has acquired information is unobservable to his rival.[9] In the bidding stage, the two bidders submit bids in a given auction format, based on the information they have acquired

[9]This is a model of *covert information acquisition*, which captures a situation where one's activity of information acquisition is not readily detectable to others, as is plausible in many cases.

in the learning stage. The model described so far is henceforth referred to as the setup I^2. We consider an alternative setup, called I^1, which is identical to I^2 except that each bidder i is initially informed of no signals and decides to learn s_i at the learning stage by incurring cost $c > 0$.[10] This setup has been studied by the previous literature, such as Persico (2000) and Shi (2012), and is used as a benchmark to compare the results from our main setup I^2.

The information each bidder i holds at the bidding stage is called bidder i's *type* and denoted by t_i, which can consist of both s_i and s_j, or only s_i, or none of the two signals, depending on the information setup as well as bidder i's learning decision in that setup. To simplify notation, for any $m, m' \in \{0, 1\}$, we let $t_i = mm'$ and $t_i = m$ indicate that bidder i is informed of $(s_i, s_j) = (m, m')$ and $s_i = m$, respectively, while $t_i = U$ indicates that bidder i is uninformed of both signals (which can only arise under I^1). We let $\overline{\Omega}_n$ denote the set of all possible types under each setup $I^n, n = 1, 2$. Then, $\overline{\Omega}_1 = \{U, 0, 1\}$ and $\overline{\Omega}_2 = \{0, 1, 00, 01, 10, 11\}$. Let v_t denote the expected value of the object to each bidder conditional on his type being t:

$$\begin{aligned}
v_{11} &= \mathbb{E}[v_i(s_i, s_j)|(s_i, s_j) = (1, 1)] = 1 = 1 - v_{00}, \\
v_1 &= \mathbb{E}[v_i(s_i, s_j)|s_i = 1] = \beta + (1 - \beta)\alpha = 1 - v_0, \\
v_{10} &= \mathbb{E}[v_i(s_i, s_j)|(s_i, s_j) = (1, 0)] = \beta = 1 - v_{01}, \quad \text{and} \\
v_U &= \mathbb{E}[v_i(s_i, s_j)] = 1/2.
\end{aligned}$$

Note that $v_{00} \leq v_0 \leq v_{01} \leq v_U \leq v_{10} \leq v_1 \leq v_{11}$, where the inequalities become strict with $\beta \in (\frac{1}{2}, 1)$. Let $v(t, t')$ denote the expected value of the object to a bidder conditional on his own type being t and his rival's type being t'. Clearly, for any m, $m' \in \{0, 1\}$ and any t,

$$v(m, m') = v(mm', t) = v(t, mm') = v_{mm'}, \; v(m, U) = v_m, \; \text{and } v(U, U) = v_U.$$

Note also that

$$v(U, 0) = \mathbb{E}[v_i(s_i, s_j)|s_j = 0] = \beta(1 - \alpha) \;\text{ and }\; v(U, 1) = \mathbb{E}[v_i(s_i, s_j)|s_j = 1] = \beta\alpha + (1 - \beta).$$

We will sometimes write v_{U0} and v_{U1} to denote $v(U, 0)$ and $v(U, 1)$, respectively.

Define the *allocative surplus* as the expected value bidders receive from the object allo-

[10] Note that we use different notations for the learning cost under I^1 and I^2, to distinguish the two different types of learning.

cation, which excludes the (expected) cost of learning. In our setup of binary signals, the allocative surplus is higher if each bidder i with $s_i = 1$ is more likely to win the object against the rival with $s_j = 0$. The maximum allocative surplus is achieved when the former always wins against the latter, and equals $\frac{1}{2}\alpha + (1-\alpha)\beta$. The *total surplus* is equal to the allocative surplus minus the (expected) cost of learning.

Throughout the paper, we focus on the symmetric sequential equilibrium—henceforth referred to as symmetric equilibrium or more simply equilibrium—, allowing for mixed strategies. The equilibrium strategy consists of learning strategy and bidding strategy. Under I^1, the equilibrium learning strategy is represented by π_U, the probability that each uninformed bidder i learns s_i. Similarly, under I^2, the equilibrium learning strategy is represented by π_0 and π_1, the probabilities that each bidder i learns s_j, $j \neq i$, conditional on his prior signal being $s_i = 0$ and 1, respectively. Given the learning strategies under a given setup I^n, we let $\Omega \subseteq \overline{\Omega}_n$ denote the set of all bidder types that arise with positive probability in equilibrium.[11] The equilibrium bidding strategy is represented by a profile of bid distributions $\{H_t\}_{t\in\Omega}$, where $H_t(b)$ is the probability that a type-t bidder submits a bid less than or equal to $b \in \mathbb{R}_+$. Let E_t denote the support of the equilibrium bid distribution H_t with $int(E_t)$ denoting its interior, and let $\overline{b}_t := \sup E_t$ and $\underline{b}_t := \inf E_t$. The equilibrium payoff for each type in the bidding stage is denoted by Γ_t. Note that this payoff does not account for the learning cost.

A sequential equilibrium requires bidders behave optimally in both learning and bidding stages. First, the learning strategy must be optimal, comparing the learning cost against the benefit of learning, which is the payoff increase that accrues in the bidding stage from learning.[12] Specifically, bidders learning with a positive probability implies the cost is no greater than the benefit, while bidders strictly randomizing the learning strategy implies the cost and benefit are equal. In the bidding stage, each bidder must bid optimally given his information (or type), his belief on the rival's type, and the rival's bidding strategy. By definition of sequential equilibrium, we impose the consistency requirement on the bidders' belief. This requirement is only slightly stronger than imposing the Bayes rule alone, in that it requires each bidder to believe that his rival follows the equilibrium learning and bidding

[11] For instance, if $\pi_U \in (0,1)$ under I^1, then $\Omega = \{U, 0, 1\}$. Likely, if $\pi_0 = 0$ and $\pi_1 \in (0,1)$ under I^2, then $\Omega = \{0, 1, 10, 11\}$.

[12] Because of the assumption that the learning decision is unobservable to the other bidder, learning an additional information can never hurt the bidder in the bidding state since he can simply ignore it. Kim (2008) shows, however, that a bidder can get worse off with learning an additional information (even without any learning cost) when the learning decision is observable, since it can cause a rival's adverse response.

strategies even after the bidder himself deviates from the equilibrium strategy in the learning stage.

3 Analysis of Setup I^1

In this section, we study the first-price and second-price auctions under the benchmark setup I^1. All proofs in this section are provided in Supplementary Material.

We first characterize a unique symmetric equilibrium for the second-price auction. In the characterization, we focus on the range of cost c that permits bidders to learn with a positive probability.[13]

Proposition 1 (Second-Price Auction). *Suppose that $c \in (0, \bar{c})$, where $\bar{c} := \frac{v_1 - v_U}{2}$. Then, there exists a unique symmetric equilibrium of the second-price auction under I^1 in which*

(i) $\pi_U = \frac{v_1 - v_U - 2c}{v_1 - v_U} \in (0, 1)$, which is increasing in α and β while decreasing in c;

(ii) each bidder of type $t \in \Omega = \{U, 0, 1\}$ bids $b_0 = v_{00} < b_U = v_U < b_1 = v_{11}$;

(iii) the payoff for each bidder is equal to $\Gamma_U = \frac{1}{2}\pi_U(v_{U0} - v_{00})$, which is decreasing in c.

To explain Part (ii) first, an uninformed bidder bids the ex ante expected value of the object, while each bidder who is informed of his own signal bids what would be the object value if his rival had the same signal. This strategy is identical to the well known equilibrium characterization for the second-price auction in the standard interdependent values setup—e.g., Milgrom and Weber (1982)—where bidders' information is exogenously given. To explain Part (i), observe from Part (ii) that each bidder i's learning yields a positive gain only when he learns $s_i = 1$ while his rival is uniformed and bids v_U in equilibrium, since if the rival is informed of $s_j = 0$ or 1, bidder i obtains the same payoff whether or not he is informed. Thus, the benefit from learning is proportional to $v_1 - v_U = \beta + (1 - \beta)\alpha - \frac{1}{2}$, capturing the informational advantage. This benefit increases as α or β increases—that is, signals become more correlated or values become more private—, explaining the effect of these parameters on π_U. Also, π_U is increasing as the learning cost c decreases, as intuitively clear. Lastly, the indifference between learning and not learning implies that the equilib-

[13] One can easily check that if $c = 0$, then $\pi_U = 1$, and if $c \geq \bar{c}$, then $\pi_U = 0$ in the unique equilibrium in both second-price and first-price auctions. It is well known that there are asymmetric equilibria in the (symmetric) second-price auction without information acquisition. In our model, there also exist asymmetric equilibria, for instance, the one where bidder i learns s_i and bids v_{11} if $s_i = 1$ and v_{00} if $s_i = 0$, and bidder $j \neq i$ bids v_U without learning s_j.

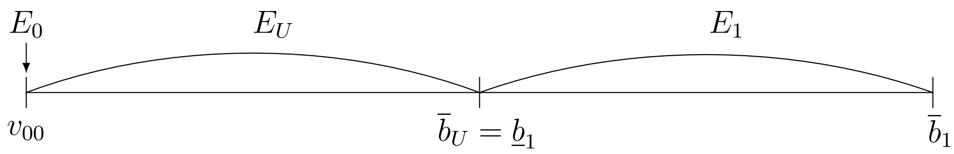


Figure 1: **Bid supports of the first-price auction under I^1 when $\pi_U > 0$.**

rium payoff of each bidder i equals Γ_U, the payoff of an uninformed bidder. This explains Part (*iii*).

We next characterize the symmetric equilibrium for the first-price auction:

Proposition 2 (First-Price Auction)**.** *Suppose that $c \in (0, \bar{c})$. Then, there exists a unique equilibrium of the first-price auction under I^1 in which*

(i) $\pi_U \in (0, 1)$ solves the equation

$$(1 - \pi_U)(1 - \alpha\pi_U) = (v_1 - 2c - \pi_U v_1)(2 - \pi_U) \tag{1}$$

and is increasing in α and β while decreasing in c;

(ii) $v_{00} = \underline{b}_0 = \bar{b}_0 = \underline{b}_U < \bar{b}_U = \underline{b}_1 < \bar{b}_1$ (refer to Figure 1), where

$$\underline{b}_1 = \frac{(1 - \pi_U)v_1 - 2c}{1 - \alpha\pi_U} \quad \text{and} \quad \bar{b}_1 = v_1 - 2c - \pi_U v_{U0}; \tag{2}$$

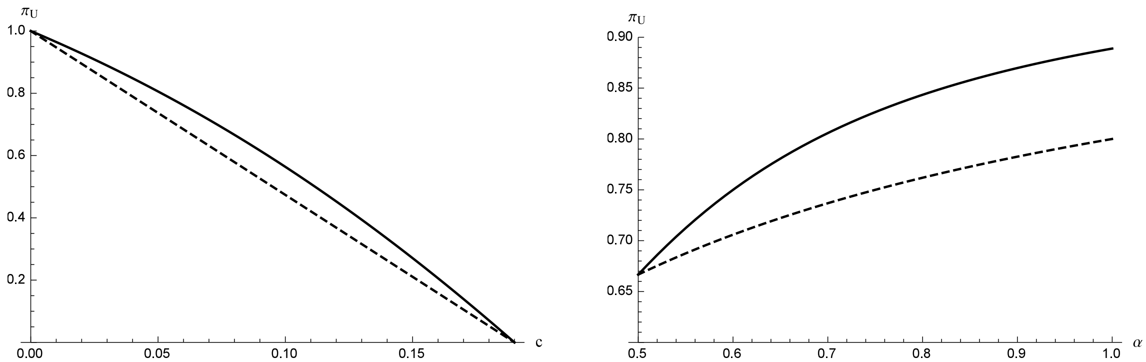
(iii) the payoff for each bidder is equal to $\Gamma_U = \frac{1}{2}\pi_U(v_{U0} - v_{00})$, which is decreasing in c.

The equilibrium characterized in Proposition 2 is similar to that of the second-price auction in Proposition 1, except that bidders are randomizing their bidding strategies as well as learning strategies. Figure 1 depicts the support of bid distribution for each type of bidders, E_t for $t \in \Omega \equiv \{U, 0, 1\}$, as given in Part (*ii*).

We now compare the equilibrium outcomes of the first-price and second-price auctions.

Proposition 3. *Under I^1, the learning probability, the bidders' payoff, and the allocative surplus are higher in the first-price auction than in the second-price auction, while the total surplus and the seller's revenue are higher in the second-price auction than in the first-price auction.*

The first-price auction induces bidders to learn their (own) signals with higher probability than does the second-price auction, as depicted in Figure 2(a), while the learning probabilities are strictly positive under both auctions. Intuitively, the higher probability of bidders



(a) Learning probability (according to c) (b) Learning probability (according to α)

Figure 2: **Comparison of learning probabilities.** Primitive values: $\alpha = 0.7$, $\beta = 0.6$, $c = 0.05$. The solid and the dashed lines represent the first- and the second-price auctions, respectively.

learning their private information (or their own signals) in the first-price auction translates into more information rent of bidders, or higher equilibrium payoff in the first-price auction (even after accounting for the learning cost).

The higher learning probability in the first-price auction is reminiscent of the result established by Persico (2000), showing (in the setup with continuum signals) that bidders acquire more accurate signals in the first-price auction than in the second-price auction. Intuitively, this is driven by the strategic advantage that matters more in the first-price auction, as explained in the introduction. Note that this advantage becomes greater when signals are more correlated, because learning one's own signal then conveys more accurate information about the other's (correlated) signal and thus his bidding strategy. This explains why the discrepancy between the learning probabilities in the two auction formats increases as α increases, as shown in Figure 2(b).

To compare the total surplus in the two auctions, recall that the total surplus is equal to the allocative surplus minus the sum of the two bidders' (expected) learning cost. The latter cost $2\pi_U c$ is proportional to the learning probability. The allocative surplus also increases in the learning probability, because when bidders are informed of their signals with higher probability, each bidder i with $s_i = 1$ is more likely to win against the rival bidder j with $s_j = 0$. Thus, both the learning cost and the allocative surplus increase going from the second-price to first-price auction, while the former dominates the latter so the total surplus is higher in the second-price auction. Lastly, the observations made so far imply that the seller's revenue, which equals the total surplus minus bidders' payoff, is higher in the second-price auction than in the first-price auction.

4 Analysis of Main Setup I^2

We now turn to the analysis of I^2 in which each bidder i is initially informed of s_i and decides whether or not to learn s_j, $j \neq i$. In this setup, we ask how bidders with different prior signals learn their rival's signal, and how it affects their bidding strategy, and thereby their payoffs, the seller's revenue, and the total surplus in the two auction formats. As we will show, the answers to these questions depend on the magnitude of learning cost (i.e., k) and the degrees of signal correlation and value interdependence (i.e., α and β). Our intuition behind the results will come from understanding a combined effect of the informational and strategic advantages on bidders' incentive to learn their rival's signal.

4.1 Second-Price Auction

The following theorem shows that the second-price auction induces no learning.

Theorem 1. *In the second-price auction under I^2, there exists a unique symmetric equilibrium in which*

(i) $\pi_1 = \pi_0 = 0$;

(ii) each bidder of type $t \in \Omega = \{0, 1\}$ *bids* $b_0 = v_{00} < b_1 = v_{11}$;

(iii) each bidder's payoff is $\frac{1}{2}(1-\alpha)(v_{10} - v_{00})$, *which is decreasing in* α, *increasing in* β *and independent of* k.

Proof. See Appendix A.1. ■

With no bidder learning the rival's signal, the equilibrium bidding strategy is identical to that of the standard setup without the learning possibility. The property of this strategy is that the winning bidder's payment is weakly lower than his (true) value, while the winning bid is weakly higher than the losing bidder's (true) value.[14] This implies, in contrast with I^1 case, that neither the winning bidder nor the losing bidder can gain from learning the other's information. Hence, $\pi_1 = \pi_0 = 0$ constitutes an equilibrium. The proof of Theorem 1 is mostly devoted to establishing the equilibrium uniqueness within the symmetric class.

[14] For instance, a bidder i with $s_i = 1$ has value v_{11} or v_{10} when the rival bidder has $s_j = 1$ or $s_j = 0$, respectively, in which case bidder i pays $b_1 = v_{11}$ (conditional upon winning) or $b_0 = v_{00}$.

Note also that given the equilibrium learning and bidding strategies, each bidder i with $s_i = 1$ wins for sure against the rival with $s_j = 0$. This implies that the allocation is fully efficient, achieving the maximum allocative surplus $\frac{1}{2}\alpha + (1-\alpha)\beta$. The total surplus also achieves the first-best, since no bidder incurs the learning cost.

4.2 First-Price Auction

Turning to the analysis of the first-price auction under I^2, the following proposition provides the overall pattern of information acquisition in equilibrium:

Proposition 4. *In the first-price auction under I^2, the following results hold:*

(i) There exists a unique symmetric equilibrium with $\pi_1 = \pi_0 = 0$ if and only if $k \geq \overline{k}_1 := \alpha(1-\alpha)(v_{11} - v_{00})$;

(ii) $\pi_0, \pi_1 < 1$ in any symmetric equilibrium;

(iii) There is no symmetric equilibrium with $\pi_1 = 0 < \pi_0$, so π_1 must be positive if $k < \overline{k}_1$.

Proof. See Appendix B.1. ■

Note that no bidder chooses to learn his rival's signal if the learning cost is above the threshold $\overline{k}_1$. With the learning cost below this threshold, a strong bidder—i.e., bidder with high prior signal—is learning with positive probability, while a weak bidder—i.e., bidder with low prior signal—may not. It suggests that strong bidders are more prone to learn their rival's signal. Indeed, as we will see later, for values of k lower than $\overline{k}_1$, strong bidders are learning with higher probability than weak bidders, while the latter may not learn at all. This is because the strategic advantage from learning rival's signal gives strong bidders a greater benefit, which comes from being able to shade their bids against a weak rival. This benefit decreases as signals become more correlated (that is, the rival is less likely to be weak). So the threshold learning cost $\overline{k}_1 = \alpha(1-\alpha)(v_{11} - v_{00})$ goes down as α increases.

We proceed with a more detailed analysis of the equilibrium in the case that learning occurs with positive probabilities (i.e., at least one of π_1 and π_0 is positive). In all equilibrium characterizations below, the support of bid distribution for each type of bidder is a connected interval: that is, $E_t = [\underline{b}_t, \overline{b}_t]$ for all $t \in \Omega$, while the interval may be degenerate (i.e., $\underline{b}_t = \overline{b}_t$). We will describe the equilibrium bidding strategies by only specifying the upper and lower bounds of the bid supports. The equilibrium bid distribution, $H_t(\cdot)$, can then be derived in a straightforward manner using the fact that the payoff for each type $t \in \Omega$ remains constant over the interval E_t.

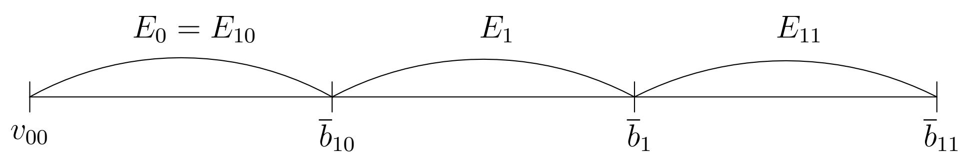


Figure 3: **Bid supports of the first-price auction under I^2 when $\pi_1 > 0 = \pi_0$**

We first provide a characterization of symmetric equilibrium in which only strong bidders learn with positive probability. In this case, a weak bidder will always be of type $t = 0$, but a strong bidder can be of type $t = 1$, $t = 10$ or $t = 11$ when he does not learn the rival's signal, learns that it is low, or learns that it is high, respectively. We thus have $\Omega = \{0, 1, 10, 11\}$.

Proposition 5. *In any symmetric equilibrium with $\pi_1 > 0 = \pi_0$ for the first-price auction under I^2, the following results hold:*

(i) π_1 solves the equation

$$\frac{v_{11} - v_{01}}{k} = \frac{1}{1-\alpha} + \frac{1}{\alpha(1-\pi_1)} - \frac{\alpha v_{01}}{k[\alpha + (1-\alpha)\pi_1]}; \tag{3}$$

(ii) $v_{00} = \underline{b}_0 = \underline{b}_{10} < \bar{b}_0 = \bar{b}_{10} = \underline{b}_1 < \bar{b}_1 = \underline{b}_{11} < \bar{b}_{11}$ (refer to Figure 3), where

$$\bar{b}_{10} = v_{11} - \frac{k}{(1-\pi_1)\alpha} - \frac{k}{1-\alpha}, \quad \bar{b}_1 = v_{11} - \frac{k}{(1-\pi_1)\alpha}, \quad \bar{b}_{11} = v_{11} - \frac{k}{\alpha}; \tag{4}$$

(iii) This equilibrium exists only if $k \in [\bar{k}_0, \bar{k}_1)$, where $\bar{k}_0$ is the (unique) solution of

$$k = (1-\alpha)\pi_1(v_{01} - \bar{b}_0). \tag{5}$$

Proof. See Appendix B.2. ■

The supports of equilibrium bid distributions in Part *(ii)* are depicted in Figure 3. Observe that the bid support of strong bidder shifts upward as he learns that the rival is strong (i.e., E_{11} lies above E_1), and likewise, it shifts downward as he learns that the rival is weak (i.e., E_{10} lies below E_1). This reflects the informational advantage. The less aggressive bidding of type $t = 10$ bidder is also a consequence of the strategic advantage: each strong bidder who learns that his rival is weak revises downward his inference of the rival's bidding strategy and shades his bid further.

Let Γ_t denote the equilibrium payoff of type t bidder, where $t \in \Omega$. The learning probability π_1 in Part *(i)* is chosen to make bidder i with $s_i = 1$ indifferent between learning and not learning; that is,

$$\Gamma_1 = (1-\alpha)\Gamma_{10} + \alpha\Gamma_{11} - k, \tag{6}$$

where the right hand side comes from the fact that when bidder i learns the rival's signal s_j, it will be $s_j = 0$ with probability $1-\alpha$ and $s_j = 1$ with probability α. Rearranging (6) yields the expression (3).

To understand Part (*iii*), note that if bidder i with signal $s_i = 0$ (i.e., weak bidder) deviates to learn his rival's signal s_j, then he could lower his bid to the lowest level v_{00} upon learning $s_j = 0$ or he could raise his bid to $\bar{b}_{10}$ upon learning $s_j = 1$, which results in the deviation payoff equal to the right hand side of (5).[15] If $k < \bar{k}_0$, then this payoff exceeds the learning cost, so the equilibrium where only strong bidders are learning cannot be sustained.

We next provide a characterization of equilibrium in which both strong and weak bidders learn with positive probabilities. Note that we have $\Omega = \overline{\Omega}_2 = \{0, 1, 00, 01, 10, 11\}$.

Proposition 6. *In any symmetric equilibrium with* $0 < \pi_0, \pi_1 < 1$ *for the first-price auction under* I^2*, the following results hold:*

(i) π_1 *solves the equation*

$$\frac{v_{11} - v_{01}}{k} = \frac{1}{1-\alpha} + \frac{1}{\alpha(1-\pi_1)} - \frac{1}{(1-\alpha)\pi_1}, \tag{7}$$

while

$$\pi_0 = \frac{v_{01} - \frac{k}{(1-\alpha)\pi_1} - \frac{k}{\alpha}}{v_{10} - \frac{k}{\alpha}} < \pi_1; \tag{8}$$

(ii) $v_{00} = \underline{b}_{00} = \bar{b}_{00} = \underline{b}_0 = \underline{b}_{10} < \bar{b}_0 = \underline{b}_{01} < \bar{b}_{01} = \bar{b}_{10} = \underline{b}_1 < \bar{b}_1 = \underline{b}_{11} < \bar{b}_{11}$ *(refer to Figure 4), where*

$$\bar{b}_0 = \frac{k}{\alpha},\ \bar{b}_{10} = v_{01} - \frac{k}{(1-\alpha)\pi_1},\ \bar{b}_1 = v_{11} - \frac{k}{\alpha(1-\pi_1)},\ \bar{b}_{11} = v_{11} - \frac{k}{\alpha}; \tag{9}$$

(iii) This equilibrium exists only if $k < \bar{k}_0$*, where* k_0 *is defined by* (5)*.*

Proof. See Appendix B.3. ■

Figure 4 illustrates the supports of equilibrium bid distributions. Note that E_{11} lies above E_1 and E_{10} lies below E_1, as was the case with $\pi_1 > 0 = \pi_0$. With the learning cost below

[15] This is the best deviation payoff, so the equilibrium sustains as long as this payoff does not exceed the learning cost.

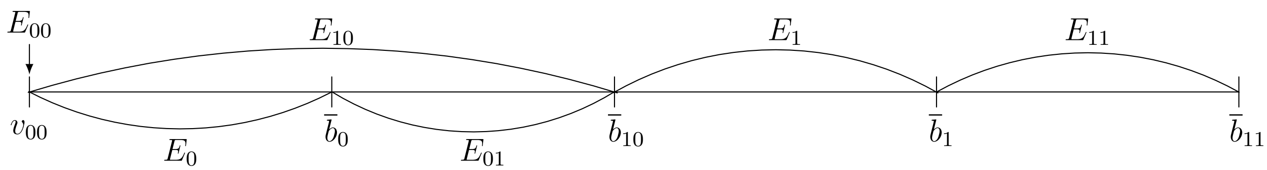


Figure 4: **Bid supports of the first-price auction under I^2 when $\pi_1, \pi_0 > 0$**

the threshold $\bar{k}_0$, weak bidders also learn with positive probability, i.e., $\pi_0 > 0$. They then adopt a more (resp., less) aggressive bidding strategy when their rival's signal turns out to be high (resp., low)—i.e., E_{01} (resp., E_{00}) lies above (resp., below) E_0. This again reflects the informational advantage. It is worth noting that the support E_{01} overlaps with the upper segment of the support E_{10}, suggesting that the type $t = 01$ tends to bid more aggressively than the type $t = 10$.[16] This is because type $t = 10$ bidder entertains the possibility of competing against a weak rival of type $t = 0$ or $t = 01$, while type $t = 01$ bidder is certain about his rival being strong and of type $t = 10$ or $t = 1$.

The following theorem summarizes the equilibrium characterization in Proposition 4 through Proposition 6 and establishes the existence of (unique) symmetric equilibrium.

Theorem 2. *Under I^2, there exists a unique symmetric equilibrium of the first-price auction in which*

(i) for $k \geq \bar{k}_1$, $\pi_1 = \pi_0 = 0$;

(ii) for $k \in [\bar{k}_0, \bar{k}_1)$, $\pi_0 = 0$ and $\pi_1 \in (0, 1)$ is given as the solution of (3), which is decreasing in k and increasing in β;

(iii) for $k < \bar{k}_0$, $\pi_1 \in (0, 1)$ is given as the solution of (7), which is decreasing in k and increasing in α and β, while $\pi_0 \in (0, 1)$ is given as (8) and decreasing in k, α and β.

(iv) each bidder's payoff is $\frac{1}{2}(1-\alpha)(v_{10}-\bar{b}_{10})$, which is increasing in k and β and decreasing in α for $k < \bar{k}_1$.[17]

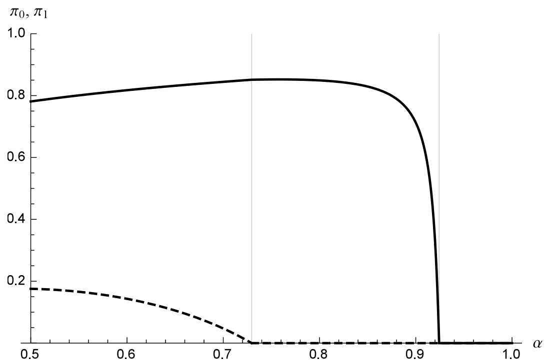
Proof. See Appendix B.4. ■

It is intuitive that the learning probabilities decrease in k.[18] To understand the effect of α

[16]In fact, a numerical analysis shows that the bid distribution of $t = 01$ first-order stochastically dominates that of $t = 10$ for certain parameter values.

[17]The term $\bar{b}_{10}$ is given by (4) for $k \in [\bar{k}_0, \bar{k}_1)$ and (9) for $k < \bar{k}_0$. It is straightforward to check that in the case $k \geq \bar{k}_1$, the bidders' payoff is increasing in β and decreasing in α while being constant in k.

[18]Interestingly, as k converges to zero, π_1 converges to 1 but π_0 converges to $\frac{v_{01}}{v_{10}} < 1$, as can be seen from (8). In fact, the limit of bidding strategies combined with these learning probabilities constitutes an equilibrium when the learning cost is zero. However, there are other equilibria with zero learning cost. Our result thus provides a selection of equilibrium at $k = 0$ that is robust to perturbation of the learning cost.



(a) Changes of π_0 and π_1 according to α (b) Changes of π_0 and π_1 according to β

Figure 5: **Learning probabilities in the first-price auction under I^2.** Primitive values: $\alpha = \beta = 0.6$ and $k = 0.07$. In panel (a), $k < \overline{k}_0$ for $\alpha < 0.73$; $k \in [\overline{k}_0, \overline{k}_1)$ for $\alpha \in [0.73, 0.92)$; $k \geq \overline{k}_1$ for $\alpha \in [0.92, 1)$. In panel (b), $k < \overline{k}_0$ for $\beta < 0.67$; $k \in [\overline{k}_0, \overline{k})$ for $\beta \in [0.67, 1]$. The solid and the dashed lines respectively represent π_1 and π_0.

on the learning probabilities, recall that under I^1, a higher α—i.e., higher correlation between signals—allows one to make more accurate inference of the other's signal and bidding strategy by learning his own signal, which gives a greater incentive to learn the latter signal. Under I^2, however, the higher correlation means that the prior signal each bidder initially holds is already more informative of his rival's signal, so bidders expect less informational or strategic gain from learning their rival's signal. While π_0 is decreasing in α as a consequence, it gives strong bidders an incentive to sustain, or even slightly increase, their learning probability as α increases (even in the range of α where $\pi_0 = 0$) unless α is too high, as depicted in Figure 5(a). With (relatively) higher α, a weak bidder learns with lower probability and also believes his rival is more likely to be weak, thereby bidding less aggressively against a strong rival, which gives the latter a greater incentive to learn the former's signal and shade his bid.[19]

The benefit from the strategic advantage depends also on the degree of value interdependence: a higher β—i.e., lower interdependence—increases the value discrepancy between strong and weak bidders, which enables strong bidders to shade their bids more and thereby draw more benefit from learning that their rival is weak. This explains why the learning probability of strong bidders, π_1, is increasing in β for any $k < \overline{k}_1$. In contrast, π_0 is negatively affected by higher β.[20] This follows from the fact that the lower interdependence

[19] It is true that with higher α, a strong bidder also believes his rival to be of the same type, which affects his learning incentive negatively. However, the overall effect turns out to be slightly positive (unless α is too high), as can be seen from Figure 5(a).

[20] It can be shown that $\pi_0 = 0$ when β is sufficiently high.

makes learning the other's signal less valuable for one's value estimation—that is, it reduces the informational advantage—while a weak bidder derives the benefit of learning mostly from the informational advantage. With π_0 being lower due to higher β, weak bidders are likely to bid less competitively when facing strong bidders, which will reinforce the learning incentive of strong bidders. Figure 5(b) depicts how the learning probabilities change with the value interdependence.

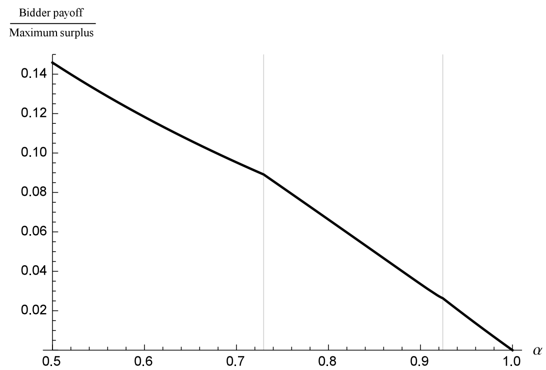
The equilibrium payoff in Part (*iv*) of Theorem 2 follows from the fact that each bidder obtains a positive payoff only when his own signal is high and his rival's signal is low, which yields the (ex-ante) payoff $\frac{1}{2}(1-\alpha)\Gamma_{10} = \frac{1}{2}(1-\alpha)(v_{10} - \bar{b}_{10})$ since, by bidding $\bar{b}_{10}$, he wins against the rival with low signal for sure (irrespective of whether the latter has learned or not). The fact that this payoff is increasing in k means that a lower learning cost (i.e., lower k) is harmful to bidders' payoff, which is intuitive since a lower cost induces both strong and weak bidders to learn with higher probability, thereby intensifying the bidding competition.

Note also that the increase in signal correlation (i.e., higher α) reduces bidders' payoff, though it induces less learning by weak bidders. This is because a higher signal correlation in itself has the effect of intensifying the bidding competition. For instance, if signals are perfectly correlated, then the entire rent for bidders will be competed away. In contrast, a higher β increases bidders' payoff through its opposed effects on weak and strong bidders' learning incentives that facilitate the bid shading by the latter bidders. The effect of α and β in Part (*iv*) should be taken with some caution since the maximum surplus, $\frac{1}{2}\alpha + (1-\alpha)\beta$, also varies with those parameters. However, their effect on the normalized payoff, which is defined as the bidders' payoff divided by the maximum surplus, remains qualitatively the same as in Part (*iv*) of Theorem 2, as depicted in Figure 6.

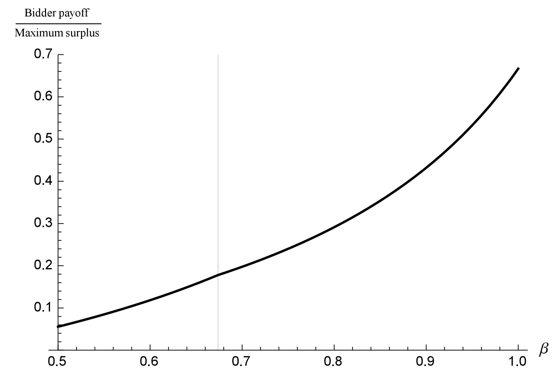
Remark 1. So far we have assumed that bidders learn their rival's signal perfectly whenever they pay the learning cost. The model can be easily extended to the case of imperfect learning. To do so, assume that when each bidder i decides to learn the rival's signal, he learns s_j, $j \neq i$, with probability $q \in (0, 1]$ but learns nothing with the remaining probability.[21] Note that q measures the precision of learning, and that $q = 1$ in our setup I^2.

Consider the case in which only strong bidders decide to learn with a positive probability.

[21] One can consider an alternative extension in which each bidder observes another signal that is *imperfectly correlated* to the rival's signal. This model is impenetrable to our analysis, however.



(a) Normalized bidder payoff according to α



(b) Normalized bidder payoff according to β

Figure 6: **Bidder's normalized payoff in the first-price auction under I^2.** Primitive values: $\alpha = \beta = 0.6$ and $k = 0.07$.

The analysis is modified only in so far as the indifference condition (6) changes to

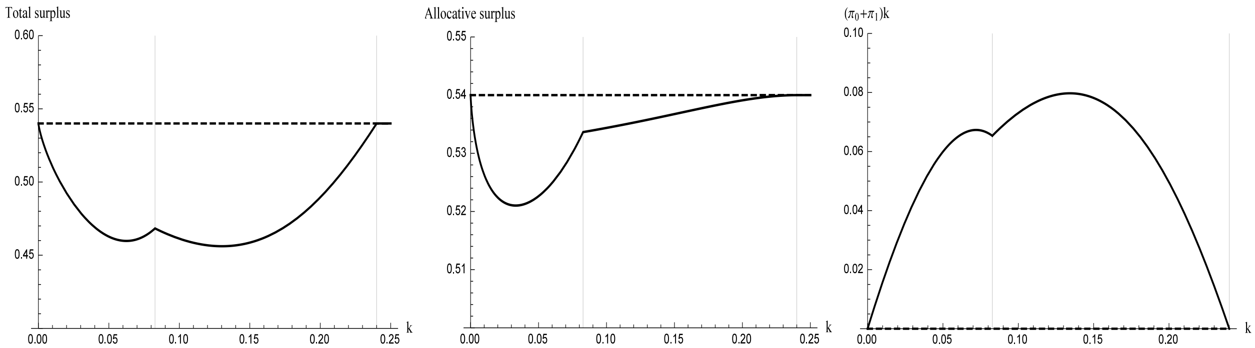
$$\Gamma_1 = q\big((1-\alpha)\Gamma_{10} + \alpha\Gamma_{11}\big) + (1-q)\Gamma_1 - k, \tag{10}$$

where the left hand side is the payoff from no learning and the right hand side is the payoff a bidder i with high prior signal expects from deciding to learn s_j. This expression follows from the fact that the learning succeeds with probability q, in which case the payoff of the bidder with high prior signal is equal to Γ_{10} and Γ_{11} with probability $1-\alpha$ and α, respectively, while, if the learning fails with probability $1-q$, then his payoff equals Γ_1. Rearranging the terms in (10), we have

$$\Gamma_1 = (1-\alpha)\Gamma_{10} + \alpha\Gamma_{11} - \frac{k}{q}.$$

Comparing this with (6) reveals that the bidder is now paying a higher cost $k/q > k$ to have the same information that he would have obtained if the learning was perfect. It is also straightforward to see that all other equilibrium conditions remain unchanged, except that π_1 is replaced by $q\pi_1$. As a consequence of these observations and Theorem 2, the *effective* learning probability, $q\pi_1$, is increasing in the learning precision q.[22] An analogous analysis applies to the case in which both strong and weak bidders decide to learn with positive probabilities.

[22] However, our numerical analysis shows that the probability of learning decision, π_1, is changing non-monotonically in q.



(a) Total surplus (b) Allocative surplus (c) Learning costs

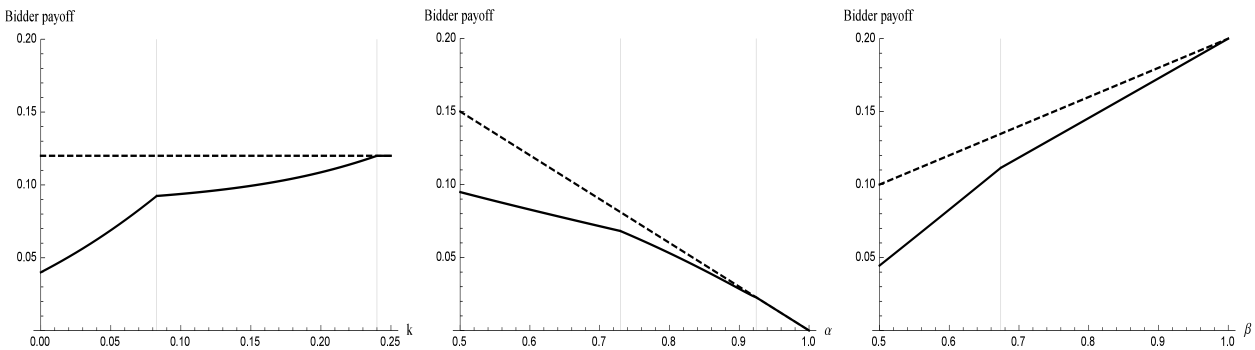
Figure 7: **Comparison of the total surplus** Primitive values: $\alpha = \beta = 0.6$. The solid and the dashed lines represent the first-price and the second-price auctions, respectively.

5 Comparison of the Two Auction Formats

Using the equilibrium characterization obtained so far under I^2, we compare the performance of the first-price and second-price auctions in terms of total surplus, bidders' payoff and the seller's revenue. To do so, let T_{FPA} and T_{SPA} denote the total surplus in the unique symmetric equilibrium of the first-price and second-price auctions, respectively. Likewise let B_{FPA} and B_{SPA} denote the bidders' equilibrium payoffs, and let R_{FPA} and R_{SPA} denote the seller's equilibrium revenues. Note that for $k \geq \overline{k}_1$, there is no learning in the equilibrium of both auctions, which leads to the outcomes with $T_{FPA} = T_{SPA}$, $B_{FPA} = B_{SPA}$, and $R_{FPA} = R_{SPA}$. Henceforth, we focus on the case where $k < \overline{k}_1$.

☐ **Total surplus.** Recall that the total surplus—or simply referred to as surplus—is equal to the allocative surplus minus the learning costs. Recall also from Theorem 1 that the second-price auction achieves the highest possible surplus for two reasons: (i) the allocation is efficient; and (ii) the learning cost is not incurred. The dashed lines in Figure 7 depicts total surplus and the allocative surplus in the second-price auction. The first-price auction, however, fails both (i) and (ii). In particular, (i) fails since strong bidders often lose to weak bidders, as can be seen from the fact that the support E_{10} overlaps with E_0 or E_{01}. The solid lines in Figure 7 depict the total surplus, the allocative surplus, and the learning costs in the first-price auction.

Proposition 7. *For any $k < \overline{k}_1$, $T_{FPA} < T_{SPA}$.*



(a) According to k (b) According to α (c) According to β

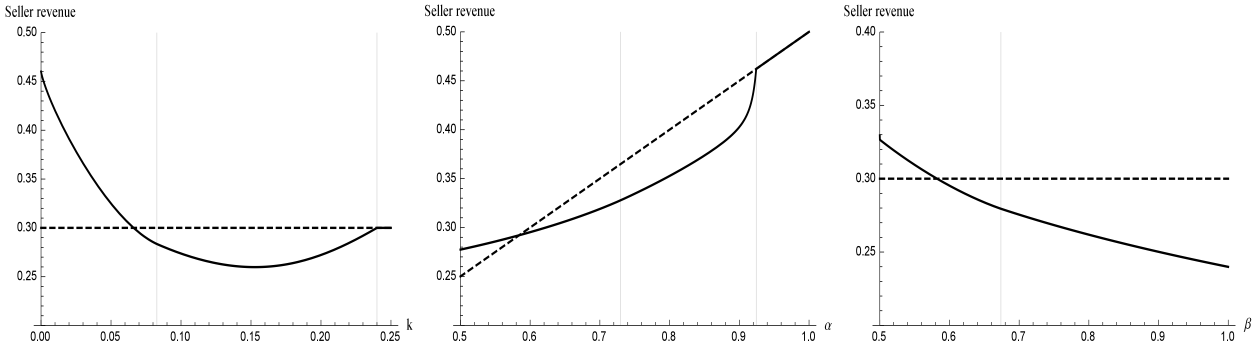
Figure 8: **Comparison of the bidders' payoff.** Primitive values: $\alpha = \beta = 0.6$ and $k = 0.07$. The solid and the dashed lines represent B_{FPA} and B_{SPA}, respectively.

□ **Bidders' payoff.** In both auction formats, each bidder obtains a positive payoff only when his (prior) signal is high and his rival's signal is low. The resulting equilibrium payoff for each bidder is $\frac{1}{2}(1-\alpha)(v_{10} - v_{00})$ in the second-price auction and $\frac{1}{2}(1-\alpha)(v_{10} - \bar{b}_{10})$ in the first-price auction, as shown by Theorem 1 and Theorem 2, respectively. Given that $\bar{b}_{10} > v_{00} = 0$, the bidders' payoff is higher in the second-price auction than in the first-price auction.

In addition, Figure 8 reveals that the difference in the bidders' payoff between the two auctions becomes larger as k, α or β becomes smaller. Recall that with smaller k, α or β, weak bidders learn their rival's signal with a higher probability. The direct effect of this learning on the bidding strategy of weak bidders themselves is ambiguous: they will bid more or less aggressively as their rival turns out to be strong or weak, respectively. However, it has an indirect effect of making strong bidders bid more aggressively, since they expect their weak rival to be informed of their high signal and thus bid more aggressively (with higher probability). It thus decreases the payoff of strong bidder facing a weak rival, which causes the bidders' equilibrium payoff in the first-price auction to decrease as well, since each bidder obtains a positive payoff only when the bidder himself is strong while his rival is weak. Indeed, the following proposition shows that the payoff difference between the two auctions is widening as k, α or β becomes smaller.

Proposition 8. *For any $k < \bar{k}_1$, $B_{SPA} > B_{FPA}$ and $B_{SPA} - B_{FPA}$ is decreasing in k, α and β.*

Proof. See Appendix C.1. ■



(a) According to k (b) According to α (c) According to β

Figure 9: **Comparison of the seller's revenue.** Primitive values: $\alpha = \beta = 0.6$ and $k = 0.07$. The solid and the dashed lines represent R_{FPA} and R_{SPA}, respectively.

☐ **Seller's revenue.** Note that the seller's revenue is equal to the total surplus minus the sum of two bidders' payoffs. As both the total surplus and the bidders' payoff are higher in the second-price auction than in the first-price auction, the revenue ranking between the two auctions can go either way. Indeed, the following proposition shows that the seller's revenue in the first-price auction is higher than that in the second-price auction when either k is small or both α and β are small, while the ranking is reversed when β is sufficiently large.

Proposition 9. *For any $k < \overline{k}_1$, the following results hold true:*

(i) $R_{FPA} > R_{SPA}$ *if* k *is close to 0, while* R_{FPA} *and* $R_{FPA} - R_{SPA}$ *are maximized at* $k = 0$[23]*;*

(ii) $R_{FPA} > R_{SPA}$ *if* α *and* β *are close to* $\frac{1}{2}$*;*

(iii) $R_{FPA} < R_{SPA}$ *if* β *is close to 1.*

Proof. See Appendix C.2. ■

To understand Parts (i) and (ii), recall from Part (iii) of Theorem 2 that the weak bidder's learning probability is decreasing in the parameter values (k, α, and β). Recall also that the weak bidder's learning has a positive effect on the strong bidder's bidding strategy and thus on the seller's revenue in the first-price auction. Moreover, according to Part (i), the seller's revenue from the first-price auction is maximized at zero learning cost (with other parameters being fixed). In this case, strong bidders learn their rival's signal at zero cost and outbid weak rival with probability one, which implies the total surplus achieves its first-best.[24] On the other hand, the bidders' payoff is minimized at $k = 0$ according to Part

[23]Since we assume $k > 0$, this should be understood as a limit result with k converging to zero.
[24]See Figure 7(a) for an illustration of this result.

(iv) of Theorem 2. Thus, the seller's revenue, which equals the total surplus minus bidders' payoffs, is maximized at $k = 0$.

In contrast, with β close to 1 (i.e, values being almost private), weak bidders never learn and then make very low bids since their value is low (close to 0). This induces strong bidders to learn with high probability as long as the learning cost is not too high (i.e., $k \leq \overline{k}_1$). Upon learning that their rival is weak, strong bidders can win the object at very low price, which is detrimental to the seller's revenue, reversing the revenue ranking in Part (iii). In fact, our numerical analysis, as in Figure 9(c), shows that there is a threshold level of value interdependence such that the first-price auction is revenue-superior to second-price auction if and only if β is small (that is, value interdependence is strong). This result is consistent with the finding by Fang and Morris (2006) that in the private values case, the first-price auction is revenue-inferior to second-price auction when bidders observe signals correlated with their rival's value, although the signals are given exogenously unlike our model.

6 Concluding Remarks

This paper investigates the problem of endogenous information acquisition in interdependent value auctions. We characterize the unique symmetric equilibrium in both first-price and second-price auctions and analyze bidders' learning and bidding behavior through two channels—informational and strategic advantages. We show that under I^2, the total surplus and the bidder payoff are higher in the second-price auction, but the ranking of the seller's revenue between the two auction formats depends on the magnitude of learning cost as well as the degrees of signal correlation and value interdependence. These findings are distinguished from the findings in the previous literature, which has mostly studied I^1 and found that the total surplus and the seller revenue are higher in the second-price auction, while the bidder payoff is higher in the first-price auction.

Appendix

We provide proofs for the second-price auction and then those for the first-price auction. All of omitted proofs are provided in Supplementary Material.

A Proofs for Section 4.1

We first provide a couple of lemmas to prove Theorem 1. Let us introduce a couple of notations. For any $t, t' \in \overline{\Omega}$, let $p(t'|t)$ be the probability with which each bidder of type t believes his rival to be of type t', given the equilibrium learning strategy. Let

$$\Omega_t := \{t' \in \Omega \,|\, p(t'|t) > 0\}$$

be the set of all rival types that a bidder of type t faces with positive probabilities. These notations will continue to be used in the analysis of the first-price auction.

Lemma 1. *Under I^n, $n = 1, 2$, if $t \in \{U, 0, 1\} \cap \Omega$, then $\underline{b}_t \geq \underline{v}(t) := \min_{t' \in \Omega_t} v(t, t')$ and $\overline{b}_t \leq \overline{v}(t) := \max_{t' \in \Omega_t} v(t, t')$ in any symmetric equilibrium of the second-price auction.*

Lemma 2. *In any symmetric equilibrium of the second-price auction under I^2, the following results hold:*
(i) If $t = mm \in \Omega$ with $m = 0$ or 1, then $E_t = \{v_t\}$;
(ii) If $\pi_1 > 0$, then $\pi_0 > 0$ and $\overline{b}_{01} > v_{01}$, while $E_t \cap (v_{01}, \overline{b}_{01}) = \emptyset$ for any $t \in \Omega_{01}$;
(iii) If $\pi_0 > 0$ and $1 \in \Omega$, then $E_1 \cap [\overline{b}_{01}, v_{11}) = \emptyset$;
(iv) If $\pi_1 = 0$, then $E_0 = \{v_{00}\}$.

A.1 Proof of Theorem 1

Proof of Part (i). To show $\pi_1 = 0$, suppose for a contradiction that $\pi_1 > 0$. Let us first consider the case that $\pi_1 \in (0, 1)$ so $1 \in \Omega$. In this case, $\pi_0 > 0$ by Part (ii) of Lemma 2, and $\underline{b}_1 \geq \underline{v}(1) = v_{10}$ by Lemma 1. Also, $E_1 \cap (v_{01}, v_{11}) = \emptyset$ from Parts (ii) and (iii) of Lemma 2. Hence, it must be the case that $E_1 \subset \{v_{01}, v_{11}\}$. If $v_{01} \in E_1$ so that type $t = 1$ puts a mass at v_{01}, then the same type can profitably deviate to bid $v_{01} + \varepsilon$ for sufficiently small $\varepsilon > 0$. Assume thus that $E_1 = \{v_{11}\} = E_{11}$, where the second equality follows from Part (i) of Lemma 2. This means that each bidder i with $s_i = 1$ can never earn a positive payoff if $s_j = 1$, which implies that it is also optimal for him to bid v_{10} irrespective of s_j. Then, he can do better by not learning s_j and bidding v_{10}, since it saves the information acquisition cost k. A similar contradiction can be established in the case $\pi_1 = 1$.

We now show $\pi_0 = 0$. Consider a bidder i with $s_i = 0$ and suppose he learns s_j. If $s_j = 0$, then he obtains zero payoff in the bidding stage, clearly. If $s_j = 1$, then the rival must be of type $t = 1$, given the fact that $\pi_1 = 0$. Since $\underline{b}_1 = \overline{b}_1 = v_1 \geq v_{01}$ by Lemma 1, bidder i can never earn a positive payoff. So, bidding v_{00} without learning s_j is better for bidder i than learning s_j, since it saves the information acquisition cost. ∎

Proof of Parts (ii) and (iii). The proof that each bidder of type $t = m \in \{0, 1\}$ must bid v_{mm} in symmetric equilibrium is similar to the proof of Part (ii) of Proposition 1 (see Supplementary Material) and hence omitted. Part (iii) immediately follows from Part (i). ∎

B Proofs for Section 4.2

To analyze the first-price auction, observe first that for any $t \in \overline{\Omega}$ and $b \geq 0$,

$$\Gamma_t(b) = \sum_{t' \in \Omega_t} p(t'|t) \left[H_{t'}(b_-) + \frac{H_{t'}(b) - H_{t'}(b_-)}{2} \right] (v(t, t') - b),$$

where $H_{t'}(b_-) := \lim_{b' \nearrow b} H_{t'}(b')$. The expression in the square bracket is due to the assumption that any bid tie is broken randomly. Note that the above payoff does not account for the learning cost. Note also that $\Gamma_t = \Gamma_t(b)$ for $b \in E_t$ (recall that Γ_t is the equilibrium payoff for type t).[25]

Lemma 3. *Call a subset $\Omega' \subset \Omega$ a **component** of Ω if $\Omega_t \subset \Omega'$ for any $t \in \Omega'$, i.e. types in Ω' face each other and no others. Then, for any component Ω' of Ω, there exists at least one type $t \in \Omega'$ with $\Gamma_t = 0$.*

Lemma 4. *Define for any $t \in \Omega$, $L_t := \{t' \,|\, \underline{b}_t \geq \bar{b}_{t'}\}$. Consider any type t deviating to bid $b \in E_{t'} \backslash E_t$ with $t' \in \Omega_t \backslash \{t\}$ such that no type puts a mass at b and there is only one type $t'' \in \Omega_t \cap \Omega_{t'}$ with $b \in E_{t''}$. Then, $\Gamma_t(b)$ is nonincreasing at such b if*

$$\frac{p(L_{t'}|t)}{p(t''|t)} \geq \frac{p(L_{t'}|t')}{p(t''|t')} \quad \text{and} \quad v(t, t'') \leq v(t', t''). \tag{B.1}$$

Also, $\Gamma_t(b)$ is nondecreasing at such b if

$$\frac{p(L_{t'}|t)}{p(t''|t)} \leq \frac{p(L_{t'}|t')}{p(t''|t')} \quad \text{and} \quad v(t, t'') \geq v(t', t''). \tag{B.2}$$

[25] To be precise, $\Gamma_t = \Gamma_t(b)$ for some $b \in int(E_t)$ or a mass point b of the distribution H_t. This is because some bid in E_t, for instance $\underline{b}_t$, can be suboptimal for type t (though $\underline{b}_t \in E_t$), in particular if there is some other type who puts a mass at $\underline{b}_t$.

B.1 Proof of Proposition 4

Proof of Part (i). Suppose that $\pi_0 = \pi_1 = 0$ in equilibrium. The existing literature, for instance Campbell and Levin (2000), shows that in this case, there is a unique equilibrium bidding strategy in which each type-0 bidder bids v_{00} for sure, while each type-1 bidder randomizes his bid over interval $[v_{00}, \bar{b}_1]$ with $\bar{b}_1 = \alpha v_{11} + (1-\alpha)v_{00}$, following the distribution

$$H_1(b) = \frac{(1-\alpha)(b - v_{00})}{\alpha(v_{11} - b)}. \tag{B.3}$$

The equilibrium payoffs for type-0 and type-1 are respectively equal to 0 and $(1-\alpha)(v_{10} - v_{00})$.

We prove the first statement by showing that no bidder has an incentive to learn his rival's signal if and only if $k \geq \bar{k}_1$. If bidder i with $s_i = 1$ deviates to learn s_j, then the maximum payoff from this deviation, exclusive of the learning cost, is given as

$$\tilde{\Gamma} = (1-\alpha)(v_{10} - v_{00}) + \max_{b \in [v_{00}, \bar{b}_1]} \alpha H_1(b)(v_{11} - b), \tag{B.4}$$

where the first term is the payoff from bidding $v_{00} + \varepsilon$ (for an arbitrary small $\varepsilon > 0$) after learning $s_j = 0$ while the second term is the payoff from bidding the optimal $b \in [v_{00}, \bar{b}_1]$ after learning $s_j = 1$. By substituting (B.3) into (B.4), we obtain

$$\tilde{\Gamma} = (1-\alpha)(v_{10} - v_{00}) + \max_{b \in [v_{00}, \bar{b}_1]} (1-\alpha)(b - v_{00}) = (1-\alpha)(v_{10} - v_{00}) + (1-\alpha)(\bar{b}_1 - v_{00}).$$

Thus, bidder i with $s_i = 1$ has no incentive to deviate and learn s_j if and only if

$$\tilde{\Gamma} - k \leq (1-\alpha)(v_{10} - v_{00}) \;\Leftrightarrow\; k \geq \alpha(1-\alpha)(v_{11} - v_{00}) = \bar{k}_1.$$

Similarly, each bidder i with $s_i = 0$ has no incentive to deviate if $k \geq \bar{k}_1$. ■

Proof of Part (ii). Suppose $\pi_1 = 1$ for a contradiction. Then, the singleton set $\{11\}$ is a component, so that $\Gamma_{11} = 0$ by Lemma 3. Thus, the payoff for each bidder i with s_i from learning s_j equals $\alpha\Gamma_{11} + (1-\alpha)\Gamma_{10} - k = (1-\alpha)\Gamma_{10} - k$. However, if he bids some $b \in E_{10}$ without learning, then the resulting payoff would be at least $(1-\alpha)\Gamma_{10} > (1-\alpha)\Gamma_{10} - k$, a contradiction. Next, suppose $\pi_0 = 1$ for a contradiction. Then, the singleton set $\{00\}$ is a component, so $\Gamma_{00} = 0$ by Lemma 3. We argue that $\Gamma_{01} = 0$, which will establish the desired contradiction since it means that bidder i with $s_i = 0$, after learning s_j, would earn zero payoff in the bidding stage, so learning only entails the cost k. If $\Gamma_{01} > 0$ to the contrary, then we must have $\bar{b}_{01} < v_{01}$, which in turn implies $\Gamma_1, \Gamma_{10} > 0$ since the type-1 bidder can get a positive payoff by bidding some $b \in (\bar{b}_{01}, v_{01})$ against his rival with zero signal. By the above observation, we cannot have $\Gamma_{11} = 0$, so $\Gamma_{11} > 0$. In sum, $\Gamma_t > 0$ for all $t \in \Omega' = \{1, 01, 10, 11\}$, which cannot hold true due to Lemma 3, however, since Ω' is a component if $\pi_0 = 1$. ■

Proof of Part (iii). Suppose for contradiction that $\pi_0 > 0 = \pi_1$. We then have $\Omega_{01} = \{1\}$, implying that $\Gamma_{01} = H_1(\bar{b}_{01})(v_{01} - \bar{b}_{01})$. Thus, the payoff of each bidder i with $s_i = 0$ from learning s_j is

$$\alpha\Gamma_{00} + (1-\alpha)\Gamma_{01} - k = (1-\alpha)\Gamma_{01} - k = (1-\alpha)H_1(\bar{b}_{01})(v_{01} - \bar{b}_{01}) - k = \Gamma_0 \geq 0, \quad \text{(B.5)}$$

where the first equality holds since $\Gamma_{00} = 0$ while the last equality holds since $\pi_0 \in (0,1)$ means that bidder i with $s_i = 0$ is indifferent between learning and not learning. Next, we must have $\underline{b}_1 \leq \underline{b}_{01}$, since otherwise $\Gamma_{01} = 0$. Since this implies that the type $t = 1$ always loses to the rival type $t = 01$ by bidding $\underline{b}_1$, we must have $\Gamma_1 = (1-\alpha)(1-\pi_0)H_0(\underline{b}_1)(v_{10} - \underline{b}_1)$. Consider now bidder i with $s_i = 1$ deviating to learn s_j. If he bids $\underline{b}_1$ after learning $s_j = 0$ and $\bar{b}_{01}$ after learning $s_j = 1$, then the resulting payoff, exclusive of the learning cost, is

$$(1-\alpha)H_0(\underline{b}_1)(v_{10} - \underline{b}_1) + \alpha H_1(\bar{b}_{01})(v_{11} - \bar{b}_{01}) = \Gamma_1 + \alpha H_1(\bar{b}_{01})(v_{11} - \bar{b}_{01}).$$

So, the net gain from the deviation is at least

$$\left[\Gamma_1 + \alpha H_1(\bar{b}_{01})(v_{11} - \bar{b}_{01}) - k\right] - \Gamma_1 = \alpha H_1(\bar{b}_{01})(v_{11} - \bar{b}_{01}) - k > 0,$$

where the inequality follows from (B.5) and the facts that $\alpha > 1 - \alpha$ and $v_{11} > v_{01}$. This means that bidder i with $s_i = 1$ has a strict incentive to learn s_j, a contradiction. ■

B.2 Proof of Proposition 5

We first provide some characterizations of symmetric equilibrium with $\pi_1 > 0$:

Lemma 5. *In any symmetric equilibrium with $\pi_1 > 0$ (whether or not $\pi_0 = 0$),*
(i) $\bigcup_{t\in\Omega} E_t$ is a connected interval while no type $t \neq 00$ puts a mass at any $b \in \left(\bigcup_{t\in\Omega} E_t\right) \setminus \{v_{00}\}$;
(ii) $E_{m0} \cap E_{m1} = \emptyset$ for $m = 0$ or 1;

(iii) $\Gamma_t > 0 = \Gamma_0 = \Gamma_{00}$ *for all* $t \neq 0, 00$*;*

(iv) $E_t \subseteq E^t := \bigcup_{t' \in \Omega_t} E_{t'}$ *for any* $t \in \Omega$*;*

(v) $\underline{b}_{00} = \overline{b}_{00} = \underline{b}_0 = v_{00}$*;*

(vi) $v_{00} < \underline{b}_1 < \underline{b}_{11}$*;*

(vii) $\underline{b}_0 = v_{00} < \overline{b}_0$*;*

(viii) $\underline{b}_{10} = v_{00} < \overline{b}_{10} = \underline{b}_1$ *while* $E_{10} = [\underline{b}_{10}, \overline{b}_{10}]$*;*

(ix) $\Gamma_1 = (1-\alpha)\Gamma_{10}$ *and* $k = \alpha\Gamma_{11}$*.*

Lemma 6. *In any symmetric equilibrium with* $\pi_1 > 0 = \pi_0$*,*

(i) $\overline{b}_0 = \overline{b}_{10} = \underline{b}_1$ *while* $E_0 = E_{10} = [v_{00}, \overline{b}_0]$*;*

(ii) $\overline{b}_1 = \underline{b}_{11}$ *while* $E_1 = [\underline{b}_1, \overline{b}_1]$ *and* $E_{11} = [\underline{b}_{11}, \overline{b}_{11}]$*.*

Proof of Parts (i) and (ii). Lemma 5 and Lemma 6 together imply that the supports of the equilibrium bids distributions must look like those in Figure 3. Given this, one can write the equilibrium conditions as follows:

$$0 = \Gamma_0(\overline{b}_0) = \alpha(v_{00} - \overline{b}_0) + (1-\alpha)\pi_1(v_{01} - \overline{b}_0) \tag{B.6}$$

$$(1-\alpha)(v_{10} - \overline{b}_0) = \Gamma_1(\overline{b}_0) = \Gamma_1(\overline{b}_1) = (1-\alpha)(v_{10} - \overline{b}_1) + \alpha(1-\pi_1)(v_{11} - \overline{b}_1) \tag{B.7}$$

$$(1-\pi_1)(v_{11} - \overline{b}_1) = \Gamma_{11}(\overline{b}_1) = \Gamma_{11}(\overline{b}_{11}) = v_{11} - \overline{b}_{11} \tag{B.8}$$

$$k = \alpha\Gamma_{11}(\overline{b}_{11}) = \alpha(v_{11} - \overline{b}_{11}), \tag{B.9}$$

where the first equalities of (B.6) and (B.9) hold due to Parts (*iv*) and (*ix*) of Lemma 5, respectively. From (B.9), $\overline{b}_{11} = v_{11} - \frac{k}{\alpha}$. Substituting this into (B.8) yields $\overline{b}_1 = v_{11} - \frac{k}{(1-\pi_1)\alpha}$, which can then be substituted into (B.7) to yield $\overline{b}_0 = v_{11} - \frac{k}{(1-\pi_1)\alpha} - \frac{k}{1-\alpha}$. We thus obtain (4). To obtain (3), rearrange (B.6) to get

$$\overline{b}_0 = \frac{\alpha v_{00} + (1-\alpha)\pi_1 v_{01}}{\alpha + (1-\alpha)\pi_1} = \frac{(1-\alpha)\pi_1 v_{01}}{\alpha + (1-\alpha)\pi_1}. \tag{B.10}$$

Equating this with $\overline{b}_0$ in (4) yields (3). It is straightforward to check that the RHS of (3) is increasing in π_1 so there exists a unique solution (if any) that solves (3). ■

Proof of Part (iii). We show that there is some $\overline{k}_0 < \overline{k}_1$ such that if $k \notin [\overline{k}_0, \overline{k}_1)$, there is no equilibrium with $\pi_1 > 0 = \pi_0$. First, one can easily check that for $k = \overline{k}_1 = \alpha(1-\alpha)$, $\pi_1 = 0$ is the (unique) solution of (3). Thus, there is no positive solution to (3) if $k \geq \overline{k}_1$, since the RHS of (3) is increasing in π_1. Next, we show that if $k < \overline{k}_0$, then each bidder i

with $s_i = 0$ can profitably deviate to learn s_j. To see the payoff from this deviation, after learning $s_j = 0$, the bidder i of type $t_i = 00$ can only obtain zero payoff (by bidding v_{00}). After learning $s_j = 1$ (with probability $1 - \alpha$), the bidder i of type $t_i = 01$ can bid $\bar{b}_0$ to obtain $(v_{01} - \bar{b}_0)$. Thus, the deviation payoff is at least $(1-\alpha)\pi_1(v_{01} - \bar{b}_0)$, which is equal to $\alpha(\bar{b}_0 - v_{00})$ by (B.6). This payoff is decreasing in k since $\bar{b}_0$ is decreasing in k.[26] This implies that the deviation is profitable for $k < \bar{k}_0$, given the definition of $\bar{k}_0$ in (5). ■

B.3 Proof of Proposition 6

Let us first provide further characterizations of symmetric equilibrium with $\pi_1, \pi_0 > 0$.

Lemma 7. *In any symmetric equilibrium with $\pi_1, \pi_0 > 0$,*
(i) $\bar{b}_0 = \underline{b}_{01}$ while $E_0 = [v_{00}, \bar{b}_0]$ and $E_{01} = [\underline{b}_{01}, \bar{b}_{01}]$;
(ii) $\bar{b}_1 = \underline{b}_{11}$ while $E_{11} = [\underline{b}_{11}, \bar{b}_{11}]$;
(iii) $\bar{b}_0 \leq \underline{b}_1$;
(iv) $\bar{b}_{01} \in [\underline{b}_1, \bar{b}_1]$ while $E_{01} = [\underline{b}_{01}, \bar{b}_{01}]$;
(v) $(1-\alpha)\Gamma_{01} = k$.

Lemma 8. *If $k \geq \bar{k}_0$, then there is no symmetric equilibrium with $\pi_1, \pi_0 > 0$.*

Lemma 9. *If $k < \bar{k}_0$, then $\bar{b}_{01} = \bar{b}_{10}$ in any symmetric equilibrium with $\pi_1, \pi_0 > 0$.*

Proof of Parts (i) and (ii). By Lemma 5, Lemma 7, and Lemma 9, the supports of the equilibrium bid distributions must look like those in Figure 4. Using this, we can write the equilibrium conditions as follows:

$$0 = \Gamma_0(\bar{b}_0) = \alpha(v_{00} - \bar{b}_0) + (1-\alpha)\pi_1 H_{10}(\bar{b}_0)(v_{01} - \bar{b}_0) \tag{B.11}$$

$$(1-\alpha)(v_{10} - \bar{b}_{10}) = \Gamma_1(\bar{b}_{10}) = \Gamma_1(\bar{b}_1) = (1-\alpha)(v_{10} - \bar{b}_1) + \alpha(1-\pi_1)(v_{11} - \bar{b}_1) \tag{B.12}$$

$$\pi_1 H_{10}(\underline{b}_{01})(v_{01} - \underline{b}_{01}) = \Gamma_{01}(\underline{b}_{01}) = \Gamma_{01}(\bar{b}_{01}) = \pi_1(v_{01} - \bar{b}_{01}) \tag{B.13}$$

$$(v_{10} - \bar{b}_{10}) = \Gamma_{10}(\bar{b}_{10}) = \Gamma_{10}(\bar{b}_0) = (1-\pi_0)(v_{10} - \bar{b}_0) \tag{B.14}$$

$$k = (1-\alpha)\Gamma_{01}(\bar{b}_{01}) = (1-\alpha)\pi_1(v_{01} - \bar{b}_{01}) \tag{B.15}$$

$$k = \alpha\Gamma_{11}(\bar{b}_1) = \alpha(1-\pi_1)(v_{11} - \bar{b}_1) \tag{B.16}$$

$$k = \alpha\Gamma_{11}(\bar{b}_{11}) = \alpha(v_{11} - \bar{b}_{11}), \tag{B.17}$$

[26] To see it, rewrite (B.6) to get $\bar{b}_0 = \frac{(1-\alpha)\pi_1 v_{01}}{\alpha + (1-\alpha)\pi_1}$, which is decreasing in k since π_1 is decreasing in k.

where the first equalities in (B.15) to (B.17) hold due to Part (ix) of Lemma 5 and Part (v) of Lemma 7. Observe that $\bar{b}_{01}$, $\bar{b}_1$ and $\bar{b}_{11}$ in (9) are directly obtained by rearranging (B.15), (B.16), and (B.17), respectively. Next, rearranging (B.11) yields

$$\bar{b}_0 = \frac{\alpha v_{00} + (1-\alpha)\pi_1 H_{10}(\bar{b}_0) v_{01}}{\alpha + (1-\alpha)\pi_1 H_{10}(\bar{b}_0)}. \tag{B.18}$$

Note that

$$H_{10}(\bar{b}_0) = H_{10}(\underline{b}_{01}) = \frac{v_{01} - \bar{b}_{01}}{v_{01} - \bar{b}_0} = \frac{k}{(1-\alpha)\pi_1(v_{01} - \bar{b}_0)}, \tag{B.19}$$

where the first equality follows from $\bar{b}_0 = \underline{b}_{10}$, the second from (B.13), and the third from substituting the expression for $\bar{b}_{01}$ in (9). We obtain the expression of $\bar{b}_0$ in (9) by substituting (B.19) into (B.18) and then solving for $\bar{b}_0$.

Let us now obtain π_0 and π_1. For π_0, rearrange (B.14) to get

$$\pi_0 = 1 - \frac{v_{10} - \bar{b}_{10}}{v_{10} - \bar{b}_0} = \frac{\bar{b}_{10} - \bar{b}_0}{v_{10} - \bar{b}_0}. \tag{B.20}$$

Substituting this equation into the expressions for $\bar{b}_{10} = \bar{b}_{01}$ and $\bar{b}_0$ in (9) yields (8). To show that π_1 is obtained by solving (7), substitute $\bar{b}_{10} = v_{01} - \frac{k}{(1-\alpha)\pi_1}$ into (B.12) to get

$$\begin{aligned}(1-\alpha)\left(v_{10} - v_{01} + \frac{k}{(1-\alpha)\pi_1}\right) &= (1-\alpha)(v_{10} - \bar{b}_1) + \alpha(1-\pi_1)(v_{11} - \bar{b}_1) \\ &= (1-\alpha)\left(v_{10} - v_{11} + \frac{k}{\alpha(1-\pi_1)}\right) + k,\end{aligned}$$

where the second equality holds since $\bar{b}_1 = v_{11} - \frac{k}{\alpha(1-\pi_1)}$. Then, (7) is obtained by rearranging the leftmost and rightmost terms of the above equation. The RHS of (7) increases from $-\infty$ to ∞ as π_1 increases from 0 to 1 while the LHS is constant, and hence there is a unique solution $\pi_1 \in (0,1)$ to (7).

Lastly, to show $\pi_1 > \pi_0$, observe first that

$$\pi_1 - \pi_0 = \pi_1 - \frac{\bar{b}_{01} - \bar{b}_0}{v_{10} - \bar{b}_0} = \frac{v_{10} - \bar{b}_{01} - (1-\pi_1)(v_{10} - \bar{b}_0)}{v_{10} - \bar{b}_0}, \tag{B.21}$$

where the first equality follows from (B.20). Next, we use $v_{11} - v_{01} = v_{10}$ and $\bar{b}_0 = \frac{k}{\alpha}$ to rewrite (7) as

$$(1-\pi_1)(v_{10}-\bar{b}_0) = \frac{k(1-\pi_1)}{1-\alpha} - \frac{k(1-\pi_1)}{(1-\alpha)\pi_1} + \frac{k\pi_1}{\alpha}$$

Substituting this and $\bar{b}_{01} = v_{01} - \frac{k}{(1-\alpha)\pi_1}$ into the numerator of the last term in (B.21),

$$\begin{aligned}
v_{10} - \bar{b}_{01} - (1-\pi_1)(v_{10}-\bar{b}_0) &= v_{10} - v_{01} + \frac{k}{(1-\alpha)\pi_1} - \frac{k(1-\pi_1)}{1-\alpha} + \frac{k(1-\pi_1)}{(1-\alpha)\pi_1} - \frac{k\pi_1}{\alpha} \\
&= v_{10} - v_{01} + \frac{k}{(1-\alpha)\pi_1} - \frac{k}{1-\alpha} + \frac{k\pi_1}{1-\alpha} + \frac{k(1-\pi_1)}{(1-\alpha)\pi_1} - \frac{k\pi_1}{\alpha} \\
&= v_{10} - v_{01} + \frac{2k}{1-\alpha}\left(\frac{1}{\pi_1} - 1\right) + \pi_1\left(\frac{k}{1-\alpha} - \frac{k}{\alpha}\right) > 0,
\end{aligned}$$

where the inequality holds since $v_{10} > v_{01}$, $\pi_1 < 1$ and $\alpha > \frac{1}{2}$. We thus have that $\pi_1 > \pi_0$. ■

Proof of Part (iii). The result follows directly from Lemma 8. ■

B.4 Proof of Theorem 2

Proof of Part (i). By Part (*iii*) of Proposition 4, there does not exist an equilibrium with $\pi_1 = 0 < \pi_0$. Then, Parts (*iv*) of Proposition 5 and Proposition 6 together imply that bidders are learning with positive probability only if $k < \bar{k}_1$. Thus, we must have $\pi_1 = \pi_0 = 0$ if $k \geq \bar{k}_1$, in which case the uniqueness (and existence) of equilibrium follows from Proposition 4. ■

Proof of Part (ii). By Parts (*ii*) and (*iii*) of Proposition 4 and Part (*iii*) of Proposition 6, there is no equilibrium where $\pi_1 = \pi_0 = 0$ or $\pi_1 = 0 < \pi_0$ or $\pi_1, \pi_0 > 0$ if $k \in [\bar{k}_0, \bar{k}_1)$. We must thus have $\pi_1 > 0 = \pi_0$, in which case π_1 is given by (3).

We now show that π_1 is decreasing in k and increasing in β. It is immediate that π_1 is decreasing in k from the fact that the LHS of (3) is decreasing in k while the RHS is increasing in k and π_1. To show π_1 is increasing in β, rewrite (3) as

$$\frac{v_{11}}{k} = \frac{1}{1-\alpha} + \frac{1}{\alpha(1-\pi_1)} + \frac{v_{01}}{k}\left(1 - \frac{\alpha}{\alpha + (1-\alpha)\pi_1}\right).$$

With $v_{01} = 1-\beta$, the RHS of this equation is decreasing in β, which implies that π_1 is increasing in β since the RHS is increasing in π_1 while the LHS is constant.

Next, while the uniqueness of equilibrium follows from Proposition 5, it remains to show that no bidder has a profitable deviation from the equilibrium bidding or learning strategy. For no profitable deviation from the equilibrium bidding strategy, we need to prove that no bidder type $t \in \Omega$ has an incentive to deviate to place a bid in $E_{t'}$ with $t' \in \Omega_t$. As with the proof of Proposition 2 (see Supplementary Material), this proof follows directly from applying Lemma 4, and hence is omitted. For no profitable deviation from the equilibrium learning strategy, it suffices to show that each bidder i with $s_i = 0$ has no incentive to deviate to learn s_j. To do so, note that after learning $s_j = 0$, it is optimal for $t_i = 00$ to bid v_{00} and obtain zero payoff, since $00 \in \Omega$. Let $\Gamma^*_{01}(k)$ denote the payoff of $t_i = 01$, as a function of k, from bidding optimally after learning $s_j = 1$. Then, the best payoff that $t_i = 0$ can expect from learning s_j is given by $(1-\alpha)\Gamma^*_{01}(k) - k$.

Claim 1. $\Gamma^*_{01}(k)$ *is decreasing in* k*.*

Claim 2. $(1-\alpha)\Gamma^*_{01}(\overline{k}_0) = \overline{k}_0$*.*

Claim 1 and Claim 2 together imply that $(1-\alpha)\Gamma^*_{01}(k) - k \leq 0$ if $k \geq \overline{k}_0$, which means that the deviation is unprofitable. ■

Proof of Part (iii). By Parts (*ii*) and (*iii*) of Proposition 4 and Part (*iv*) of Proposition 5, there is no equilibrium in which $\pi_1 = \pi_0 = 0$ or $\pi_1 > 0 = \pi_0$ or $\pi_1 = 0 < \pi_0$, if $k < k_0$. We must thus have $\pi_1, \pi_0 > 0$ in equilibrium (if any), in which case π_1 and π_0 are given by (7) and (8), respectively.

We now prove the effects of k on π_1 and π_0. The fact that π_1 is decreasing in k is immediate from the fact that the RHS of (7) is increasing in π_1 while the LHS is increasing in k. For the effect of k on π_0, note that $\overline{b}_0$ and $\overline{b}_{10}$ in (9) are increasing and decreasing in k, respectively. Given this, the middle expression of (B.20) is decreasing in k, and so is π_0. The effects fo α and β on π_1 and π_0 follow from the next claim.

Claim 3. *For any* $k < \overline{k}_0$*,* $\partial\pi_1/\partial\alpha, \partial\pi_1/\partial\beta > 0$*, while* $\partial\pi_0/\partial\alpha, \partial\pi_0/\partial\beta < 0$*.*

Lastly, while the uniqueness of equilibrium follows from Proposition 6, we need to show that no type $t \in \Omega$ has an incentive to deviate to bid some $b \in E_{t'}$ where $t' \in \Omega_t \backslash \{t\}$.[27] Since this result follows directly from applying Lemma 4 in many cases, we only analyze the cases in which the proof relies on the following claim (whose proof is contained in Supplementary Material).

[27]Clearly, there is no profitable deviation from the equilibrium learning strategy, since each bidder is indifferent between learning and not learning irrespective of his signal.

Claim 4. *$\Gamma_0(b)$ is decreasing in $b \in E_{01}$, $\Gamma_1(b)$ is constant for $b \in E_{10}$, and $\Gamma_{01}(b)$ is increasing in $b \in E_0$.*

Consider first type $t = 0$—for whom $\Omega_t = \{0, 1, 10, 00\}$—deviating to bid some $b \in E_{10} \backslash E_0 = E_{01}$. Since $\Gamma_0(b)$ is decreasing in $b \in E_{01}$, this deviation is unprofitable. Next, consider $t = 1$—for whom $\Omega_t = \{1, 0, 01, 11\}$—deviating to bid some $b \in E_{10} = E_0 \cup E_{10}$. Since the deviation payoff $\Gamma_1(b)$ is constant across the interval E_{10}, we have $\Gamma_1(b) = \Gamma_1(\overline{b}_{10}) = \Gamma_1(\underline{b}_1) = \Gamma_1$ for all $b \in E_{10}$, so such deviation is unprofitable. Lastly, consider type $t = 01$—for whom $\Omega_t = \{1, 10\}$—deviating to bid $b \in E_0 = E_{10} \backslash E_{01}$. Since $\Gamma_{01}(b)$ is increasing in $b \in E_0$, we have $\Gamma_{01}(b) \leq \Gamma_{01}(\overline{b}_0) = \Gamma_{01}(\underline{b}_{01}) = \Gamma_{01}$ for all $b \in E_0$, as desired. ■

Proof of Part (iv). We begin with a couple of observations as follows (whose proofs are again contained in Supplementary Material):

Claim 5. $\frac{\partial(1-\alpha)(v_{10}-\overline{b}_0)}{\partial\alpha} < 0$ *for any* $k \in [\overline{k}_0, \overline{k}_1)$.

Claim 6. *For any $k < \overline{k}_1$, $\overline{b}_{10}$ is decreasing in k, α and β.*

Consider first the case $k \in [\overline{k}_0, \overline{k}_1)$. The (ex-ante) equilibrium payoff for each bidder equals

$$\tfrac{1}{2}\Gamma_0 + \tfrac{1}{2}\left[(1-\pi_U)\Gamma_1 + \pi_U(\alpha\Gamma_{11} + (1-\alpha)\Gamma_{10} - k)\right] = \tfrac{1}{2}(1-\alpha)\Gamma_{10} = \tfrac{1}{2}(1-\alpha)(v_{10} - \overline{b}_0),$$

where the first equality holds since $\Gamma_0 = 0$, $\Gamma_1 = (1-\alpha)\Gamma_{10}$, and $\alpha\Gamma_{11} = k$. Note that since $\overline{b}_0 = \overline{b}_{10}$ and $\overline{b}_{10}$ is decreasing in k and β by Claim 6, the equilibrium payoff, $\frac{1}{2}(1-\alpha)(v_{10} - \overline{b}_0)$, is increasing in k and β. The comparative statics regarding α follows from Claim 5.

Consider next the case $k < \overline{k}_0$. The (ex-ante) equilibrium payoff for each bidder equals

$$\begin{aligned}&\tfrac{1}{2}\left[(1-\pi_0)\Gamma_0 + \pi_0(\alpha\Gamma_{00} + (1-\alpha)\Gamma_{01} - k)\right] + \tfrac{1}{2}\left[(1-\pi_1)\Gamma_1 + \pi_1(\alpha\Gamma_{11} + (1-\alpha)\Gamma_{10} - k)\right] \\ &= \tfrac{1}{2}\pi_0\left((1-\alpha)\Gamma_{01} - k\right) + \tfrac{1}{2}\left[(1-\alpha)\Gamma_{10} + \pi_1(\alpha\Gamma_{11} - k)\right] = \tfrac{1}{2}(1-\alpha)\Gamma_{10} = \tfrac{1}{2}(1-\alpha)(v_{10} - \overline{b}_{10}),\end{aligned}$$

where the first equality follows from $\Gamma_0 = \Gamma_{00} = 0$ and $\Gamma_1 = (1-\alpha)\Gamma_{10}$, and the second equality from $\alpha\Gamma_{11} = k = (1-\alpha)\Gamma_{01}$. To see how the expression after the third equality changes in k and β, note that $\overline{b}_{10} = \overline{b}_{01}$ is decreasing in β and k by Claim 6. Thus, the equilibrium payoff is increasing in k and β. To see the effect of α, write the ex-ante payoff as

$$\tfrac{1}{2}(1-\alpha)\Gamma_{10} = \tfrac{1}{2}(1-\alpha)(v_{10}-\overline{b}_0) = \tfrac{1}{2}(1-\alpha)\left(v_{10} - v_{01} + \frac{k}{(1-\alpha)\pi_1}\right) = \tfrac{1}{2}(1-\alpha)(2\beta-1)+\frac{k}{2\pi_1},$$

which is decreasing in α since $2\beta - 1 \geq 0$ and π_1 is increasing in α. ■

C Proofs for Section 5

C.1 Proof of Proposition 8

The difference of bidders' payoff between the two auctions is $\frac{1}{2}(1-\alpha)\overline{b}_{10}$. Since $\overline{b}_{10}$ is decreasing in k and β by Claim 6, so is $\frac{1}{2}(1-\alpha)\overline{b}_{10}$. Next, note that $\frac{\partial(1-\alpha)\overline{b}_{10}}{\partial\alpha} = -\overline{b}_{10} + (1-\alpha)\frac{\partial\overline{b}_{10}}{\partial\alpha} < 0$, where the inequality follows from Claim 6.

C.2 Proof of Proposition 9

Proof of Part (i). We first prove that as $k \to 0$, the total surplus in the first-price auction, T_{FPA}, approaches the first-best level $\frac{1}{2}\alpha+(1-\alpha)\beta$. Since the learning cost vanishes as $k \to 0$, we only need to show that the allocative surplus approaches the first-best level, which holds true if the winning probability of type $t = 10$ against the rival of type $t = 0$ or $t = 01$ approaches 1 as $k \to 0$. To show this, it suffice to prove that for any fixed small $\varepsilon > 0$, there is sufficiently small $\overline{k}(< \overline{k}_0)$ such that for $k < \overline{k}$, $H_{01}(b') > 1-\varepsilon = 1 - H_{10}(b')$ for some $b' \in int(E_{01})$, since it will imply that the winning probability of type $t = 10$ against type $t = 0$ or $t = 01$ is at least $(1-\pi_0) + \pi_0(1-\varepsilon)^2$, which becomes arbitrarily close to 1 by making ε sufficiently small. With k close to 0 and thus smaller than $\overline{k}_0$, the bidding distributions of type $t = 01$ and $t = 10$ on E_{01} are given as

$$H_{01}(b) = \frac{(1-\pi_0)}{\pi_0}\frac{(b-\overline{b}_0)}{v_{10}-b} \quad \text{and} \quad H_{10}(b) = \frac{v_{01}-\overline{b}_{01}}{v_{01}-b}. \tag{C.1}$$

Observe also that as $k \to 0$, we have $\pi_1 \to 1$, $\pi_0 \to \frac{v_{01}}{v_{10}}$, $\overline{b}_0 \to v_{00} = 0$, and $\overline{b}_{01} = \overline{b}_{10} \to v_{01}$. Now let b' be defined such that $H_{10}(b') = \varepsilon$. By (C.1), we have $b' = \frac{\overline{b}_{01}-(1-\varepsilon)v_{01}}{\varepsilon}$, which converges to v_{01} as $k \to 0$ since $\overline{b}_{01} \to v_{01}$ as $k \to 0$. Given this and (C.1), we have $H_{01}(b') = \frac{(1-\pi_0)}{\pi_0}\frac{(b'-\overline{b}_0)}{v_{10}-b'} \to \frac{v_{10}-v_{01}}{v_{01}}\frac{v_{01}}{v_{10}-v_{01}} = 1$ as $k \to 0$ since $\pi_0 \to \frac{v_{01}}{v_{10}}$ and $\overline{b}_0 \to v_{00} = 0$ as $k \to 0$. Thus, one can find sufficiently small k such that $H_{01}(b') > 1-\varepsilon$, as desired.

To prove that $R_{FPA} > R_{SPA}$ at $k \simeq 0$, recall from Part (*iv*) of Theorem 2 that each bidder's payoff in the first-price auction is $\frac{1}{2}(1-\alpha)(v_{10}-\bar{b}_{10}) = \frac{1}{2}(1-\alpha)\left(2\beta - 1 + \frac{k}{(1-\alpha)\pi_1}\right)$, where the equality follows from $\bar{b}_{10}$ in (9). Thus, $B_{FPA} = (1-\alpha)(2\beta - 1 + \frac{k}{(1-\alpha)\pi_1})$, which converges to $(1-\alpha)(2\beta-1)$ as $k \to 0$. Therefore, at $k \simeq 0$,

$$R_{FPA} = T_{FPA} - B_{FPA} \simeq \tfrac{1}{2}\alpha + (1-\alpha)\beta - (1-\alpha)(2\beta-1) = \tfrac{1}{2}\alpha + (1-\alpha)(1-\beta).$$

In the second-price auction, according to Theorem 1, a positive payment is made only when both bidders have high signal and equals $v_{11} = 1$, which means that $R_{SPA} = \frac{1}{2}\alpha < R_{FPA} \simeq \frac{1}{2}\alpha + (1-\alpha)(1-\beta)$ at $k \simeq 0$, as desired.

Next, by Theorem 2, the bidders' payoff decreases as k decreases. Thus, it is minimized as $k \to 0$. Combining this with the above finding that the total surplus approaches the first-best level as $k \to 0$ implies that the seller's revenue is maximized as $k \to 0$. ■

Proof of Part (ii). We show that at $\alpha = \beta = \frac{1}{2}$, $R_{FPA} > R_{SPA}$, from which the desired result will follow since the seller's revenue as well as the equilibrium strategy is continuous at $\alpha = \beta = \frac{1}{2}$. So let $\alpha = \beta = \frac{1}{2}$, and note that $R_{SPA} = \frac{1}{2}\alpha = \frac{1}{4}$ while the allocative surplus in the first-price auction is equal to $\frac{1}{2}$.[28] Consider the case of $k \in [\bar{k}_0, \bar{k}_1)$ in the first-price auction in which

$$(1-\alpha)(v_{10} - \bar{b}_0) = (1-\alpha)\left(v_{10} - \frac{(1-\alpha)\pi_1 v_{01}}{\alpha + (1-\alpha)\pi_1}\right) = \frac{1}{4(1+\pi_1)}, \tag{C.2}$$

where the first equality follows from $\bar{b}_0$ in (4) and some rearrangement, and the second equality holds since $\alpha = \beta = \frac{1}{2}$. Hence, $R_{FPA} = \frac{1}{2} - \pi_1 k - \frac{1}{4(1+\pi_1)}$, where $\pi_1 k$ is the learning cost. We thus have

$$R_{FPA} - R_{SPA} = \frac{1}{2} - \pi_1 k - \frac{1}{4(1+\pi_1)} - \frac{1}{2} = \pi_1\left(\frac{1}{4(1+\pi_1)} - k\right) = \frac{\pi_1^3}{8 + 4\pi_1(1-\pi_1)} > 0,$$

where the last equality holds since

$$k = \frac{(1-\pi_1)(2+\pi_1)}{4(2-\pi_1)(1+\pi_1)}, \tag{C.3}$$

which follows from (3) and the fact that $\alpha = \beta = \frac{1}{2}$.

[28] With $\beta = \frac{1}{2}$, the values are common across bidders and equal to $\frac{1}{2}$, which means that the allocative surplus is $\frac{1}{2}$ irrespective of the allocation, as long as someone obtains the object.

Before turning to the case of $k < \overline{k}_0$, we show that $\overline{k}_0 = \frac{1}{10}$ at $\alpha = \beta = \frac{1}{2}$. To see this, recall that from (5), $\overline{k}_0$ is the unique solution to

$$k = (1-\alpha)(v_{01} - \overline{b}_0)\pi_1 = \frac{\pi_1}{4(1+\pi_1)},$$

where the second equality follows from the fact that $v_{10} = v_{01} = \frac{1}{2}$ (since $\beta = \frac{1}{2}$) and (C.2). Equating this with (C.3), we have $\pi_1 = \frac{2}{3}$ and $\overline{k}_0 = \frac{1}{10}$. Next, for $k < \overline{k}_0$,

$$B_{FPA} = (1-\alpha)(v_{10} - \overline{b}_{01}) = (1-\alpha)\left(v_{10} - v_{01} + \frac{k}{(1-\alpha)\pi_1}\right) = \frac{k}{\pi_1},$$

where the last equality holds since $\alpha = \beta = \frac{1}{2}$. Thus,

$$\begin{aligned} R_{FPA} - R_{SPA} = T_{FPA} - B_{FPA} - R_{SPA} &= \frac{1}{2} - (\pi_0 + \pi_1)k - \frac{k}{\pi_1} - \frac{1}{4} \\ &= \frac{1}{4} - \left(1 - \frac{4k}{(1-4k)\pi_1} + \pi_1\right)k - \frac{k}{\pi_1} = \frac{2 - \pi_1 - 2\pi_1^3 + 2\pi_1^4}{4(1-2\pi_1)(1-3\pi_1+\pi_1^2)}, \end{aligned}$$

where the second equality follows from (8) and the fact that $\alpha = \beta = \frac{1}{2}$, and the last equality holds since $k = \frac{(1-\pi_1)\pi_1}{4(3\pi_1 - \pi_1^2 - 1)}$ from substituting $\alpha = \beta = \frac{1}{2}$ into (7). One can show that the numerator of the RHS of the last equality attains its minimum value 0.019 at $\pi_1 \approx 0.637$, and the denominator is strictly positive for any $\pi_1 > \frac{1}{2}$, which holds true since the fact that $k = \frac{(1-\pi_1)\pi_1}{4(3\pi_1 - \pi_1^2 - 1)} < \overline{k}_0 = \frac{1}{10}$ implies $\pi_1 > \frac{2}{3}$. ■

Proof of Part (iii). With $\beta \simeq 1$, we have $v_{01} \simeq 0$, so the expression in (8) becomes negative, meaning that we must have $\pi_0 = 0$ in the equilibrium. Thus, from (3), we obtain $\pi_1 \simeq \frac{\alpha(1-\alpha)-k}{\alpha(1-\alpha-k)} > 0$. Substituting this into $\overline{b}_0 = \overline{b}_{10}$ in (4), we have $\overline{b}_0 = \overline{b}_{10} \simeq 0$ and thus

$$B_{FPA} = 2 \times \tfrac{1}{2}(1-\alpha)(v_{10} - \overline{b}_{10}) \simeq (1-\alpha) \tag{C.4}$$

since $v_{10} \simeq 1$ with $\beta \simeq 1$. Also, $T_{FPA} \leq \frac{1}{2}\alpha + (1-\alpha) - \pi_1 k$. Hence,

$$R_{FPA} = T_{FPA} - B_{FPA} \leq \tfrac{1}{2}\alpha + (1-\alpha) - \pi_1 k - B_{FPA} \simeq \tfrac{1}{2}\alpha - \pi_1 k < \tfrac{1}{2}\alpha = R_{SPA}.$$

where the approximate equality follows from (C.4) and the strictly inequality holds since $k, \pi_1 > 0$. ■

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Supplementary Material for "Learning Rival's Information in the Interdependent Value Auctions"

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January 18, 2019

Contents

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This notes consists of two sections. Section S1 provides proofs for I^1 case in Section 3, and Section S2 provides omitted proofs in the Appendix of the paper.

S1 Proofs for Section 3

For any type-t bidder and his bids b and $\tilde{b}$, we say that b **ex-post dominates** $\tilde{b}$ **for** t (or $\tilde{b}$ is ex-post dominated by b for t) if b yields payoff no less than $\tilde{b}$ does against equilibrium strategy of any rival type $t' \in \Omega_t$ and strictly higher payoff against equilibrium strategy of some rival type $t' \in \Omega_t$. Note that no equilibrium bid can be ex-post dominated.

S1.1 Proof of Proposition 1

We first establish a useful lemma.

Lemma S1. *Under I^1, $E_U = \{v_U\}$ and $E_m = \{v_{mm}\}$ for each $m = 0, 1$, and $\pi_U \neq 0, 1$ in any symmetric equilibrium.*

Proof. We first show that $E_U = \{v_U\}$ and $E_m = \{v_{mm}\}$ for each $m = 0, 1$.[1] To begin, we use Lemma 1 to make the following observations: (a) $\underline{b}_U, \overline{b}_U \in [v_0, v_1]$; (b) $v_{00} \leq \underline{b}_0 \leq \overline{b}_0 \leq v_{01} < v_{10} \leq \underline{b}_1 \leq \overline{b}_1 \leq v_{11}$. An implication of (b) is that type-1 bidder always wins against type-0 rival. Observe now that $\overline{b}_0 \leq v_0$ since otherwise we must have some $b \in (v_0, v_{01}] \cap E_0$ that is ex-post dominated by $b' = v_0$ for type-0 bidder for the following reason: if he faces type-0 or type-U rival, then he only avoids some negative payoffs (without giving up any positive payoff) by reducing his bid from b to b';[2] he always loses against type-1 rival by bidding either b or b'. Given that $\overline{b}_0 \leq v_0$ and $\underline{b}_1 \geq v_{10} > v_U$, we must have $[v_0, v_U) \cap E_U = \emptyset$ since otherwise we can find some $b \in [v_0, v_U) \cap E_U$ that is ex-post dominated by $b' = v_U > b$ for type U. The reason is that by bidding b' instead of b, a type-U bidder can increase the probability of earning positive payoff against the same type rival while his winning status—and thus his payoff—against other types does not change. The argument so far establishes $[v_{00}, v_U) \cap E_U = \emptyset$. Given this, we must $\overline{b}_0 = v_{00}$ since any bid $b \in (v_{00}, v_{01}] \cap E_0$ can be shown to be ex-post dominated by v_{00} for type-0 bidder, using an argument similar to the above.

[1] Our argument below establishes this result in all three possible cases: $\pi_U = 0$ so $\Omega = \{U\}$; $\pi_U = 1$ so $\Omega = \{0, 1\}$; and $\pi_U \in (0, 1)$ so $\Omega = \{U, 0, 1\}$.

[2] If his rival is of type 0 (resp., type U), then this negative payoff amounts to $v_{00} - b$ (resp., $v_0 - b$), multiplied by the probability that the rival's bid is between b and b', which is strictly positive for type-0 bidder because of the assumption that $b \in (v_0, v_{01}] \cap E_0$.

In sum, we must have $E_0 = \{v_{00}\}$ and $E_U \cap [v_{00}, v_U) = \emptyset$. An analogous argument can be used to establish $E_1 = \{v_{11}\}$ and $(v_U, v_{11}] \cap E_U = \emptyset$. That $E_U \cap [v_{00}, v_U) = E_U \cap (v_U, v_{11}] = \emptyset$ means $E_U = \{v_U\}$, as desired.

Let us now show that $\pi_U \neq 0, 1$ in any symmetric equilibrium. Suppose for a contradiction that $\pi_U = 0$, in which case the equilibrium payoff is equal to 0 since both bidders bid v_U. Now consider bidder i deviating to learn s_i. Since bidder j is uninformed and thus bids v_U, bidder i's optimal response is to bid any $b < v_U$ and lose if $s_i = 0$, and to bid any $b > v_U$ and win if $s_i = 1$, which results in the expected payoff equal to $\frac{1}{2}(v_1 - v_U)$. This payoff is smaller than the learning cost c since $c < \bar{c} = \frac{v_1 - v_U}{2}$, so the deviation is profitable. Suppose for another contradiction that $\pi_U = 1$, in which case the equilibrium payoff is equal to $\frac{1}{2}(1-\alpha)(v_{10} - v_{00}) - c$.[3] Now consider bidder i deviating not to learn s_i. Being uninformed of s_i, bidder i's optimal bid is any $b \in (v_{00}, v_{11})$ such that he wins against the rival of $t_j = 0$ (who bids v_{00}) while losing against the rival of type $t_j = 1$ (who bids v_{11}), which results in the expected payoff equal to $\frac{1}{2}\big(v(U,0) - v_{00}\big)$. This is greater than the equilibrium payoff since $\frac{1}{2}\big(v(U,0) - v_{00}\big) = \frac{1}{2}(1-\alpha)\beta = \frac{1}{2}(1-\alpha)(v_{10} - v_{00})$. ■

We are now ready to prove Proposition 1.

Proof of Parts (i) and (ii). By Lemma S1, we must have $\pi_U \in (0,1)$ and $\Omega = \{U, 0, 1\}$. Also, each type $t = U$, 0, and 1 must bid v_U, v_{00}, and v_{11} with probability 1, respectively.

Let us now consider each bidder i's incentive to learn s_i, provided that his rival learns s_j with probability π_U and follows the bidding strategy given above. If bidder i is uninformed of s_i, then his expected payoff is $\Gamma_U = \frac{1}{2}\pi_U(v_{U0} - v_{00})$, since he obtains a positive payoff $v_{U0} - v_{00}$ only when his rival learns $s_j = 0$ and bids v_{00}. If bidder i is informed of s_i, then he obtains a positive payoff only when he learns $s_i = 1$ while his rival either remains uninformed and bids v_U, or learns $s_j = 0$ and bids v_{00}. The resulting expected payoff is

$$\frac{1}{2}\Gamma_1 - c = \frac{1}{2}(1-\pi_U)(v_1 - v_U) + \frac{1}{2}\pi_U(1-\alpha)(v_{10} - v_{00}) - c.$$

It is straightforward to show that if $c \in (0, \bar{c})$, then there exists $\pi_U \in (0,1)$ such that $\Gamma_U = \frac{1}{2}\Gamma_1 - c$, so each bidder is indifferent between learning and not learning. Also, solving this equation and substituting $v_{U0} = \beta(1-\alpha)$, $v_{10} = \beta$, and $v_{00} = 0$ yield $\pi_U = \frac{v_1 - v_U - 2c}{v_1 - v_U}$. ■

[3] This is because, with $\pi_U = 1$, each bidder i obtains a positive payoff only when $(s_i, s_j) = (1, 0)$ so he wins and pays v_{00}.

Proof of Part (iii). Since $\pi_U \in (0,1)$, each bidder must be indifferent between learning and not learning. Hence, each bidder's (ex-ante) payoff is $\Gamma_U = \frac{1}{2}\pi_U(v_{U0} - v_{00})$, which is decreasing in c as $\pi_U = \frac{v_1 - v_U - 2c}{v_1 - v_U}$ is decreasing in c. ■

S1.2 Proof of Proposition 2

Let π_U^i be bidder i's learning probability and E_t^i be the support of equilibrium bid distribution of bidder i with type $t \in \Omega$, where $i = 1, 2$. Also, let $\Gamma_t^i(b)$ denote the payoff of bidder i with type t from bidding $b \in E_t^i$, given that the other bidder employs an equilibrium bidding strategy. We first establish a series of lemmas.

Lemma S2. *In any equilibrium with $\pi_U^1, \pi_U^2 \in (0,1)$, the following results hold:*

(i) $E := \cup_{t\in\Omega} E_t^1 = \cup_{t\in\Omega} E_t^2$, while E is a connected interval.

(ii) E_t^i is a connected interval for all $i = 1, 2$.

(iii) $b \le b' \le b''$ for any $b \in E_0^i$, $b' \in E_U^i$ and $b'' \in E_1^i$ for all $i = 1, 2$.

(iv) $\underline{b}_0^1 = \underline{b}_0^2$ while $\Gamma_0^1 = \Gamma_0^2 = 0$.

(v) $E_1^i = E_1^j$ and $\bar{b}_U^i = \bar{b}_U^j$.

Proof. Since the proof of Part (i) is standard, we omit it. To prove Part (iii) first, let us write the payoff for bidder $i = 1, 2$ of type U and type 0 from bidding $b \in E$ as

$$\Gamma_U^i(b) = (1 - \pi_U^j)H_U^j(b)(v_U - b) + \tfrac{1}{2}\pi_U^j H_0^j(b)(v_{U0} - b) + \tfrac{1}{2}\pi_U^j H_1^j(b)(v_{U1} - b) \quad \text{and}$$
$$\Gamma_0^i(b) = (1 - \pi_U^j)H_U^j(b)(v_0 - b) + \alpha\pi_U^j H_0^j(b)(v_{00} - b) + (1-\alpha)\pi_U^j H_1^j(b)(v_{01} - b),$$

where $j \neq i$ and $j = 1, 2$. We then have

$$\begin{aligned}
&(1-\alpha)\Gamma_U^i(b) - \tfrac{1}{2}\Gamma_0^i(b) \\
&= (1 - \pi_U^j)H_U^j(b)\big((1-\alpha)v_U - \tfrac{1}{2}v_0 + (\alpha - \tfrac{1}{2})b\big) + \pi_U^j H_0^j(b)\big(\tfrac{1}{2}1 - \alpha)v_{U0} - \tfrac{1}{2}\alpha v_{00} + (\alpha - \tfrac{1}{2})b\big) \\
&\quad + \pi_U^j H_1^j(b)\big(\tfrac{1}{2}(1-\alpha)v_{U1} - \tfrac{1}{2}(1-\alpha)v_{01}\big) \\
&= (1 - \pi_U^j)H_U^j(b)\big(\tfrac{1}{2}(1-\alpha)\beta + (\alpha - \tfrac{1}{2})b\big) + \pi_U^j H_0^j(b)\big(\tfrac{1}{2}(1-\alpha)^2\beta + (\alpha - \tfrac{1}{2})b\big) \\
&\quad + \pi_U^j H_1^j(b)\tfrac{1}{2}(1-\alpha)\alpha\beta,
\end{aligned}$$

which is strictly increasing in b, since $H_U^j(b)$, $H_0^j(b)$, or $H_1^j(b)$ is strictly increasing in $b \in E$. This implies that (a) $\Gamma_U^i(b)$ is strictly increasing in b whenever $\Gamma_0^i(b)$ is weakly increasing. Similarly, we can also show that $\frac{1}{2}\Gamma_1^i(b) - \alpha\Gamma_U^i(b)$ is strictly increasing in b, which implies that

(b) $\Gamma_1^i(b)$ is strictly increasing in b whenever $\Gamma_U^i(b)$ is weakly increasing. It is straightforward to see that the statement of Part (ii) follows from combining (a) and (b).

To prove Part (*ii*), note that given Part (*iii*), each E_t^i must be a connected interval since otherwise $E = \cup_{t\in\Omega} E_t^i$ would not be a connected interval. The proof of Part (*iv*) follows from the standard argument and is thus omitted.

To see Part (*v*), note first that we must have $\overline{b}_1^i = \overline{b}_1^j =: \overline{b}_1$, since if $\overline{b}_1^i > \overline{b}_1^j$, then bidder i of type 1 could profitably decrease his bid from $\overline{b}_1^i$ to some $\overline{b}_1^i - \varepsilon \in (\overline{b}_1^j, \overline{b}_1^i)$. We now argue that $\underline{b}_1^i = \underline{b}_1^j$. Suppose for a contradiction that $\underline{b}_1^i < \underline{b}_1^j$. Observe that since $\Gamma_1^i = v_1 - \overline{b}_1^i = v_1 - \overline{b}_1^j = \Gamma_1^j$, we have

$$\begin{aligned}\Gamma_1^j(\underline{b}_1^j) &= (1-\pi_U^i)(v_1 - \underline{b}_1^j) + (1-\alpha)\pi_U^i(v_{10} - \underline{b}_1^j) + \alpha\pi_U^i H_1^i(\underline{b}_1^j)(v_{11} - \underline{b}_1^j) \\ &= (1-\pi_U^j)(v_1 - \underline{b}_1^j) + (1-\alpha)\pi_U^j(v_{10} - \underline{b}_1^j) = \Gamma_1^i(\underline{b}_1^j).\end{aligned}$$

Rearranging the LHS and RHS of the second equality, we obtain

$$\begin{aligned}&\alpha\pi_U^i H_1^i(\underline{b}_1^j)(v_{11} - \underline{b}_1^j) = (\pi_U^i - \pi_U^j)\big(v_1 - (1-\alpha)v_{10} - \alpha\underline{b}_1^j\big) \\ \Leftrightarrow\ &\alpha\pi_U^i H_1^i(\underline{b}_1^j)(v_{11} - \underline{b}_1^j) = (\pi_U^i - \pi_U^j)\alpha(v_{11} - \underline{b}_1^j) \\ \Leftrightarrow\ &H_1^i(\underline{b}_1^j) = \frac{\pi_U^i - \pi_U^j}{\pi_U^i},\end{aligned} \tag{S1}$$

where the second equivalence holds since $v_1 = \alpha v_{11} + (1-\alpha)v_{10}$. The fact that $\pi_U^i, \pi_U^j \in (0,1)$ implies

$$\Gamma_U^i = \tfrac{1}{2}\Gamma_0^i + \tfrac{1}{2}\Gamma_1^i - c = \tfrac{1}{2}\Gamma_1^i - c = \tfrac{1}{2}\Gamma_1^j - c = \tfrac{1}{2}\Gamma_0^j + \tfrac{1}{2}\Gamma_1^j - c = \Gamma_U^j, \tag{S2}$$

where the second and the fourth equalities follow from Part (*iv*). Recall that $\frac{1}{2}\Gamma_1^i(b) - \alpha\Gamma_U^i(b)$ is strictly increasing, which implies that $\Gamma_U^i(b)$ is strictly decreasing whenever $\Gamma_1^i(b)$ is constant. Given this and $\overline{b}_U^i = \underline{b}_1^i < b_U^j = \overline{b}_U^j$, we must have $\Gamma_1^i(\underline{b}_1^j) = \Gamma_1^i(\underline{b}_1^i) = \Gamma_1^i$ and $\Gamma_U^i(\underline{b}_1^j) < \Gamma(\underline{b}_1^i) = \Gamma_U^i$. Combining this with (S2), we obtain

$$\begin{aligned}&\Gamma_U^i(\underline{b}_1^j) = (1-\pi_U^j)(v_U - \underline{b}_1^j) + \tfrac{1}{2}\pi_U^j(v_{U0} - \underline{b}_1^j) \\ &< \Gamma_U^i = \Gamma_U^j = \Gamma_U^j(\underline{b}_1^j) \\ &= (1-\pi_U^i)(v_U - \underline{b}_1^j) + \pi_U^i\tfrac{1}{2}(v_{U0} - \underline{b}_1^j) + \tfrac{1}{2}\pi_U^i H_1^i(\underline{b}_1^j)(v_{U1} - \underline{b}_1^j) \\ &= (1-\pi_U^i)(v_U - \underline{b}_1^j) + \pi_U^i\tfrac{1}{2}(v_{U0} - \underline{b}_1^j) + \tfrac{1}{2}(\pi_U^i - \pi_U^j)(v_{U1} - \underline{b}_1^j),\end{aligned}$$

where the last equality follows from (S1). This inequality can be written as

$$(1-\pi_U^j)(v_U-\underline{b}_1^j)+\tfrac{1}{2}\pi_U^j(v_{U0}-\underline{b}_1^j)+\tfrac{1}{2}\pi_U^j(v_{U1}-\underline{b}_1^j)=v_U-\underline{b}_1^j$$
$$<(1-\pi_U^i)(v_U-\underline{b}_1^j)+\tfrac{1}{2}\pi_U^i(v_{U0}-\underline{b}_1^j)+\tfrac{1}{2}\pi_U^i(v_{U1}-\underline{b}_1^j)=v_U-\underline{b}_1^j,$$

where the equalities hold since $\frac{1}{2}v_{U0}+\frac{1}{2}v_{U1}=v_U$. A desired contradiction thus is obtained. Lastly, the fact that $\overline{b}_U^i=\underline{b}_1^i$ for each $i=1,2$ then implies $\overline{b}_U^i=\overline{b}_U^j$. ■

Lemma S3. *There is no asymmetric equilibrium with $\pi_U^i=1$ and $\pi_U^j=0$.*

Proof. By Part (v) of Lemma S2, we have $\underline{b}_1^i=\underline{b}_1^j$, which means $H_1^i(\underline{b}_1^j)=0$ (since neither bidder would put a mass at $\underline{b}_1^i=\underline{b}_1^j$). From this and (S1), we obtain $\pi_U^i=\pi_U^j$. This shows that there is no asymmetric equilibrium with $\pi_U^1\neq\pi_U^2$ and $\pi_U^i\in(0,1)$ for all $i=1,2$.

Next, we show that there is no equilibrium with $\pi_U^i=1$ and $\pi_U^j=0$. Suppose for a contradiction that there exists an equilibrium with $\pi_U^i=1$ and $\pi_U^j=0$. Using the argument similar to the proofs of Lemma S2, we can show that $E_U^j=E_0^i\cup E_1^i$ is a connected interval while E_1^i lies above E_0^i. Let $\underline{b}:=\underline{b}_0^i=\underline{b}_U^j$.

Note that we must have $\overline{b}_0=\underline{b}_0=v_0$. To see this, suppose $\overline{b}_0<v_0$ to the contrary. Then, both bidder j (of type U, clearly) and bidder i of type 0 earn a positive payoff since they can bid some $b\in(\overline{b}_0,v_0)$. Since $\Gamma_U^j(b)$ is the same for all $b\in E_U^j$, this implies that $\Gamma_U^j(\underline{b})=\frac{1}{2}H_0^i(\underline{b})(v_{U0}-\underline{b})$ is positive and so is $H_0^i(\underline{b})$, showing that bidder i of type 0 must put a mass on $\underline{b}$. Similarly, bidder j also puts a mass on $\underline{b}$. But then, either bidder j or bidder i of type 0 can deviate to take a mass from $\underline{b}$ and bid just above $\underline{b}$, since this yields a discrete jump in the probability of winning with only a slightly higher payment condition upon winning. For the same reason, we cannot have $\underline{b}_0<v_0$. Thus, $E_0^i=\{v_0\}$, and hence $E_U^j=E_1^i=[v_0,\overline{b}]$ for some $\overline{b}\in(v_0,v_1]$.

Now, bidder i can deviate by not learning s_j and bidding $b'\in E_1$ arbitrary close to v_0, say $b'=v_0+\varepsilon$. To see this, note that bidder i's expected payoff under such a deviation is $H_U^j(b')(v_U-b')$, whereas his expected payoff under the original strategy is

$$\tfrac{1}{2}\Gamma_1^i-c=\tfrac{1}{2}H_U^j(b')(v_1-b')-c,$$

since $\Gamma_0^i=0$ and $\Gamma_1^i=\Gamma_1^i(b')$. Hence,

$$H_U^j(b')(v_U-b')-\left(\tfrac{1}{2}H_U^j(b')(v_1-b')-c\right)=H_U^j(b')(v_U-\tfrac{1}{2}v_1-\tfrac{1}{2}b')+c=-\tfrac{1}{2}H_U(b')\varepsilon+c>0,$$

where the second equality holds since $v_U = \frac{1}{2}$, $v_0 = 1 - v_1$ and $b' = v_0 + \varepsilon$, and the last inequality holds for sufficiently small ε. ■

Lemma S4. *There is no asymmetric equilibrium.*

Proof. Thanks to Lemma S2 and Lemma S3, we must have $\pi_U^i = \pi_U^j = \pi_U$ in any equilibrium. We now prove that $\underline{b}_U^i = \underline{b}_U^j$. Suppose for a contradiction that $\underline{b}_U^i < \underline{b}_U^j$. Since $\Gamma_U^i = \Gamma_U^j$ from the proof of Lemma S2, we must have

$$\Gamma_U^i = \Gamma_U^i(\underline{b}_U^j) = \tfrac{1}{2}\pi_U(v_{U0} - \underline{b}_U^j) = (1-\pi_U)H_U^i(\underline{b}_U^j)(v_U - \underline{b}_U^j) + \tfrac{1}{2}\pi_U(v_{U0} - \underline{b}_U^j) = \Gamma_U^j(\underline{b}_U^j) = \Gamma_U^j,$$

which is a contradiction since $v_U < \underline{b}_U^j$ and $H_U^i(\underline{b}_U^j) > 0$. Given $\underline{b}_U^i = \underline{b}_U^j$, Lemma S2 implies that $E_t^i = E_t^j$ for all $t \in \Omega$ while each E_t^i is a connected interval. It follows easily from this that every equilibrium must be symmetric. ■

Lemma S5. *Suppose $c \in (0, \bar{c})$. There is no symmetric equilibrium with $\pi_U = 0$ or 1.*

Proof. Suppose that $\pi_U = 0$. Then, each uninformed bidder must bid v_U in the unique equilibrium. If bidder i deviates to learn $s_i = 1$, then he can bid $v_U + \varepsilon$ to obtain a payoff equal to $v_1 - v_U - \varepsilon$. So his expected payoff from deviation is at least $\frac{v_1 - v_U - \varepsilon}{2}$, which is greater than c if ε is sufficiently small, meaning the deviation is profitable.

Suppose that $\pi_U = 1$. The existing literature, for instance **?**, shows that the unique equilibrium in this case must have each type-0 bidder bidding v_{00} for sure and each type-1 bidder randomizing over an interval $[v_{00}, \bar{b}]$ for some $\bar{b} > v_{00}$, while the equilibrium payoff is equal to $\frac{1}{2}(1-\alpha)(v_{10} - v_{00}) - c = \frac{(1-\alpha)\beta}{2} - c$. Consider a bidder i's not to learn s_i and bid $v_{00} + \varepsilon$, which yields the expected payoff of $\frac{v_{U0} - v_{00} - \varepsilon}{2} = \frac{(1-\alpha)\beta - \varepsilon}{2}$ at least. This payoff is greater than the equilibrium payoff if ε is sufficiently small, so the deviation is profitable. ■

Lemma S4 and Lemma S5 show that any equilibrium must be symmetric and $\pi_U \in (0, 1)$ in such an equilibrium. We now prove Part (i) to Part (iii) of Proposition 2.

Proof of Parts (i) and (ii). Since $v_{00} = \underline{b}_U < \bar{b}_U = \underline{b}_1 < \bar{b}_1$, we have

$$\Gamma_U = \Gamma_U(b) = (1-\pi_U)H_U(b)(v_U - b) + \tfrac{1}{2}\pi_U(v_{U0} - b) \text{ for } b \in [\underline{b}_U, \bar{b}_U],$$
$$\Gamma_1 = \Gamma_1(b) = (1-\pi_U)(v_1 - b) + \pi_U\big((1-\alpha)(v_{10} - b) + \alpha H_1(b)(v_{11} - b)\big) \text{ for } b \in [\underline{b}_1, \bar{b}_1].$$

Since $\frac{1}{2}\Gamma_1 - c = \Gamma_U$, the two equations in (2) are derived by setting $\frac{1}{2}\Gamma_1(\underline{b}_1) - c = \Gamma_U(v_{00})$ and $\frac{1}{2}\Gamma_1(\bar{b}_1) - c = \Gamma_U(v_{00})$, respectively, and also by substituting $v_{11} = 1 - v_{00} = 1$, $v_{U0} =$

$(1-\alpha)\beta = (1-\alpha)v_{10}$, and $v_1 = \alpha v_{11} + (1-\alpha)v_{10}$. Next, by solving $\Gamma_U(v_{00}) = \Gamma_U(\underline{b}_1)$ and substituting $v_U = 1/2$ and $v_{00} = 0$, we have $\underline{b}_1 = \frac{1-\pi_U}{2-\pi_U}$. Equating this to another expression for $\underline{b}_1$ in (2) yields (1).

Now, let $f(\pi_U)$ and $g(\pi_U)$ denote the LHS and RHS of (1). Then,

$$f(1) = 0 > -2c - \beta(\alpha - \tfrac{1}{2}) = g(1).$$

It can be easily checked that g single-crosses f from above in the range $(-\infty, 1)$. Thus, a (unique) solution of (1) exists in (0,1) if and only if $f(0) < g(0)$, which is equivalent to $c < \bar{c}$. Observe that the increase in α or β makes f shift down and g shift up, which implies that f and g intersect at higher value of π_U, given the aforementioned single-crossing property. Likewise, as c increases, g shifts down and thus intersects with f at lower value of π_U. ■

Proof of Part (iii). The indifference between learning and not learning means that the equilibrium payoff for each bidder equals $\Gamma_U = \Gamma_U(v_{00}) = \frac{1}{2}\pi_U(v_{U0} - v_{00})$, which is decreasing in c as π_U is decreasing in c. ■

We complete the proof of Proposition 2 by showing that no type $t \in \Omega$ has an incentive to deviate to bid some $b \in E_{t'}$ for some $t' \in \Omega_t \backslash \{t\}$.[4] First, consider type $t = 1$ deviating to bid $b \in E_U$. To use Lemma 4, we check (B.2). To do this, set $t = 1$ and $t' = U$, so $L_{t'} = \{0\}$ and $\Omega_1 \cap \Omega_{t'} = \{U\}$ (i.e., $t'' = U$):

$$\frac{p(L_{t'}|t)}{p(t''|t)} = \frac{(1-\alpha)\pi_U}{1-\pi_U} < \frac{\frac{1}{2}\pi_U}{1-\pi_U} = \frac{p(L_{t'}|t')}{p(t''|t')},$$

and $v(t, t'') = v_1 > v_U = v(t', t'')$. Thus, (B.2) holds so $\Gamma_1(b)$ is nondecreasing in $b \in E_U$, meaning that $\Gamma_1(b) \le \Gamma_1$ for any $b \in E_U$.

Consider next type $t = U$. Clearly, it is not optimal to deviate to bid $b \in E_0 = \{v_{00}\}$. Suppose that $t = U$ deviates to bid $b \in E_1$. We check (B.1) to use Lemma 4, again. To do this, set $t = U$ and $t' = 1$, $L_{t'} = \{0, U\}$, and $\Omega_t \cap \Omega_{t'} = \{1\}$ (i.e., $t'' = 1$). Then, $v(t, t'') = v_{U1} \le v_{11} = v(t', t'')$. Also,

$$\frac{p(L_{t'}|t)}{p(t''|t)} = \frac{\frac{1}{2}\pi_U + (1-\pi_U)}{\frac{1}{2}\pi_U} > \frac{(1-\alpha)\pi_U + (1-\pi_U)}{\alpha\pi_U} = \frac{p(L_{t'}|t')}{p(t''|t')}.$$

[4] It is clear that a deviation to any $b \notin \cup_{t\in\Omega} E_t$ is unprofitable.

Thus, (B.1) holds so $\Gamma_U(b)$ is nonincreasing in $b \in E_1$ by Lemma 4, meaning that $\Gamma_U \geq \Gamma_U(b)$ for $b \in E_1$.

Lastly, consider type $t = 0$. Suppose he deviates to bid in E_U. We check (B.1) by setting $t = 0$, $t' = U$, $L'_t = \{0\}$, and $\Omega_t \cap \Omega_{t'} = \{U\}$ (i.e., $t'' = U$):

$$\frac{p(L_{t'}|t)}{p(t''|t)} = \frac{\frac{1}{2}\pi_U}{1 - \pi_U} = \frac{p(L_{t'}|t')}{p(t''|t')},$$

and $v(t, t'') = v(0, U) = v_0 < v_U = v(U, U) = v(t', t'')$. Thus, (B.1) holds so $\Gamma_0(b)$ is nonincreasing for any $b \in E_U$, meaning that $\Gamma_0 \geq \Gamma_0(b)$ for any $b \in E_U$. Suppose now that type $t = 0$ places his bid in E_1. We check (B.1) by setting $t = 0$, $t' = 1$, $L_{t'} = \{0, U\}$, and $\Omega_t \cap \Omega_{t'} = \{1\}$ (i.e., $t'' = 1$):

$$\frac{p(L_{t'}|t)}{p(t''|t)} = \frac{\alpha\pi_U + (1 - \pi_U)}{(1 - \alpha)\pi_U} > \frac{(1 - \alpha)\pi_U + (1 - \pi_U)}{\alpha\pi_U} = \frac{p(L_{t'}|t')}{p(t''|t')},$$

and $v(t, t'') = v_{01} < v_{11} = v(t', t'')$. Thus, (B.1) holds so $\Gamma_0(b)$ is nonincreasing for any $b \in E_1$, meaning that $\Gamma_0 \geq \Gamma_0(b') \geq \Gamma_0(b)$ for any $b \in E_1$ and $b' \in E_U$. In sum, there is no profitable deviation.

S1.3 Proof of Proposition 3

Let $a = I$ and $a = II$ respectively denote the first-price auction and the second-price auction. Recall that π_U^I satisfies (1) and $\pi_U^{II} = \frac{v_1 - v_U - 2c}{v_1 - v_U}$ from Proposition 1. Letting f and g denote the LHS and RHS of (1), observe that for $c \in (0, \bar{c})$,

$$f(\pi_U^{II}) - g(\pi_U^{II}) = \frac{2(2\alpha - 1)c(c - \bar{c})}{(v_1 - v_U)^2} < 0.$$

Since g single-crosses f from above over the interval $[0, 1]$ if $c \leq \bar{c}$, it follows that $\pi_U^I > \pi_U^{II}$. This implies that each bidder's payoff is higher in the first-price auction is higher than in the second-price auction, since each bidder's payoff in each auction $a = I, II$ is $\frac{1}{2}\pi_U^a(v_{U0} - v_{00})$.

Next, the total surplus in auction $a = I, II$, denoted by TS^a, is $TS^a = AS^a - 2\pi_U^a c$, where AS^a denotes the allocative surplus which is given by

$$\begin{aligned} AS^a &= (\pi_U^a)^2 \left(\tfrac{1}{2}\alpha v_{11} + (1 - \alpha)v_{10} + \tfrac{1}{2}\alpha v_{00}\right) + \pi_U^a(1 - \pi_U^a)\left(v_1 + v_{U0}\right) + (1 - \pi_U^a)^2 v_U \\ &= (\pi_U^a)^2 \left(\tfrac{1}{2}\alpha + (1 - \alpha)\beta\right) + \pi_U^a(1 - \pi_U^a)(\alpha + 2\beta - 2\alpha\beta) + \tfrac{1}{2}(1 - \pi_U^a)^2. \end{aligned}$$

We thus have

$$AS^I - AS^{II} = \big(\pi_U^I - \pi_U^{II}\big)(1-\alpha)(\beta - \tfrac{1}{2})\big(2 - (\pi_U^I + \pi_U^{II})\big) > 0,$$

where the inequality holds since $\pi_U^I + \pi_U^{II} < 2$ for $c > 0$. Using this, we have that

$$\begin{aligned} TS^{II} - TS^I &= (AS^{II} - 2\pi_U^{II}c) - (AS^I - 2\pi_U^I c) \\ &= \big(\pi_U^I - \pi_U^{II}\big)\left[2c - (1-\alpha)(\beta - \tfrac{1}{2})\big(2 - (\pi_U^I + \pi_U^{II})\big)\right]. \end{aligned}$$

To determine the sign of it, let $T(c) := l(c) - m(c)$ denote the terms in the square bracket, where $l(c) = 2c$ and $m(c) = (1-\alpha)(\beta - \frac{1}{2})\big(2 - (\pi_U^I + \pi_U^{II})\big)$. Note that both $l(c)$ and $m(c)$ are increasing in c (since π_U^I and π_U^{II} are decreasing in c). Suppose that $T(\hat{c}) = 0$ for some $\hat{c} \in (0, \overline{c})$, that is, $2\hat{c} = (1-\alpha)(\beta - \frac{1}{2})\big(2 - (\pi_U^I + \pi_U^{II})\big)$, or equivalently,

$$\pi_U^I + \pi_U^{II} = 2 - \frac{2c}{(1-\alpha)(\beta - \frac{1}{2})}. \tag{S3}$$

We then have that

$$T'(\hat{c}) = l'(\hat{c}) - m'(\hat{c}) = 2 + 2(1-\alpha)(\beta - \tfrac{1}{2})\left.\frac{\partial(\pi_U^I + \pi_U^{II})}{\partial c}\right|_{c=\hat{c}} = -2 < 0,$$

where the last equality holds since $\left.\frac{\partial(\pi_U^I + \pi_U^{II})}{\partial c}\right|_{c=\hat{c}} = -\frac{2}{(1-\alpha)(\beta - \frac{1}{2})}$ from (S3). This shows that m crosses l from below at $c = \hat{c}$. But then, we reach a contradiction since $l(0) = m(0) = 0$ (because $\pi_U^I = \pi_U^{II} = 1$ at $c = 0$) and $l(\overline{c}) = 2\overline{c} = \beta + (1-\beta)\alpha - \frac{1}{2} > m(\overline{c}) = 2(1-\alpha)(\beta - \frac{1}{2})$ (because $\pi_U^I = \pi_U^{II} = 0$ at $c = \overline{c}$). Therefore, $T(c) = 0$ only when $c = 0$ and $T(c) > 0$ for all $c \in (0, \overline{c})$. We thus have that $TS^{II} > TS^I$ for $c \in (0, \overline{c})$.

Lastly, note that the seller's revenue is the same as the total surplus minus the sum of the two bidders' payoffs. Since the total payoff and each bidder's payoff are respectively higher and lower in the second-price auction than in the first-price auction, it follows immediately that the seller's revenue is higher in the second-price auction than in the first-price auction.

S2 Omitted Proofs

S2.1 Proof of Lemma 1

Note that if $t \in \Omega \cap \{U, 0, 1\}$, then $t \in \Omega_t$. Suppose for a contradiction that $\overline{b}_t > \overline{v}(t)$, and consider any $\tilde{b} \in (\overline{v}(t), \overline{b}_t]$. Since $\overline{v}(t)$ is the highest value that type-t bidder can obtain, facing any possible rival's type in Ω_t, it is clear that bidding $\overline{v}(t)$ can do as well as bidding $\tilde{b}$ (given the second-price auction rule). Moreover, $\overline{v}(t)$ does better than $\tilde{b}$ when the rival is of the same type t since then the type-t bidder would incur a lower loss with $\overline{v}(t)$ than with $\tilde{b}$. One can use a similar argument to prove that $\underline{b}_t \geq \underline{v}(t)$.

S2.2 Proof of Lemma 2

Proof of Part (i). We prove that $\underline{b}_t = \overline{b}_t = v_t$ for $t = mm \in \Omega$. Note first that if $t = mm \in \Omega$, then $t \in \Omega_t$ in any symmetric equilibrium. If $\underline{b}_t < v_t$, then type t bidder, when his rival has the same type t, can increase the probability of earning a positive payoff by bidding v_t instead of some equilibrium bid $\tilde{b} \in E_t \cap [\underline{b}_t, v_t)$, which means v_t ex-post dominates $\tilde{b}$ for $t = mm$ (since v_t is also a weakly dominant strategy), a contradiction. A similar contradiction arises in the case $\overline{b}_t > v_t$. ■

Proof of Part (ii). Suppose for a contradiction that $\pi_0 = 0$ or $\pi_0 > 0$ while $\overline{b}_{01} \leq v_{01}$. Then, all type t' that the type $t = 1$ can face will be bidding at most $v(t', 1)$, which is (weakly) smaller than $v(1, t')$—the expected value of type $t = 1$ facing type t'. So bidding v_{11} is clearly optimal for each bidder i informed of $s_i = 1$, whether he learns s_j. By not learning, however, he can save the learning cost, k, so $\pi_1 > 0$ cannot be true. Also, if some type $t' \in \Omega_{01}$ were to place bids (with positive probability) between v_{01} and $\overline{b}_{01}$, then some equilibrium bid $\tilde{b} \in E_{01} \cap (v_{01}, \overline{b}_{01})$ would be ex-post dominated by v_{01} for $t = 01$ since bidding v_{01} is weakly dominant for $t = 01$ and the type $t = 01$ facing t' will incur a loss by bidding $\tilde{b}$. ■

Proof of Part (iii). Suppose not. Then, there must be at least two equilibrium bids $b, b' \in E_1 \cap [\overline{b}_{01}, v_{11})$ for $t = 1$ such that $H_1(b') > H_1(b)$ since the type $t = 1$ cannot put a mass at any bid in $E_1 \cap [\overline{b}_{01}, v_{11})$ in equilibrium. Given that all types t', except $t' = 01$, that the type $t = 1$ can face will be bidding at most $v(1, t')$, b' is better than b for the type $t = 1$ since it gives him a higher probability of winning and obtaining positive payoffs against the same rival type. ■

Proof of Part (iv). First, note that Ω_0 can possibly include $0, 1$, and 00. From Lemma 1, $\underline{b}_1 \geq \underline{v}(1) = v(1,0) = v_{10}$ while $E_{00} = \{v_{00}\}$ by Part (i). These facts imply that the type $t = 0$, whatever amount he bids, can never obtain a positive payoff when facing the rival type $t' = 1, 00$. On the other hand, if $\overline{b}_0 > v_{00}$, then the type $t = 0$, bidding some equilibrium bid $\tilde{b} > v_{00}$, will incur a loss when facing the same rival type. So v_{00} is the only optimal bid available for $t = 0$ in any symmetric equilibrium. ∎

S2.3 Proof of Lemma 3

Suppose to the contrary that $\Gamma_t > 0$ for all $t \in \Omega'$. Consider types $t \in \Omega'$ for which $\underline{b}_t = \underline{b} := \min_{t' \in \Omega'} \underline{b}_{t'}$. In order for these types to have positive payoffs, there must be some type that puts a mass at $\underline{b}$. (Otherwise, one of the types bidding $\underline{b}$ must earn zero payoff.) Then, such a type can deviate to bid slight above $\underline{b}$ and increase its payoff by a discrete amount.

S2.4 Proof of Lemma 4

We only prove the first statement since then the second statement follows immediately from reversing the inequalities. Note first that for $b \in E_{t'}$, we can write

$$\Gamma_{t'}(b) = p(t''|t')H_{t''}(b)(v(t', t'') - b) + \sum_{\hat{t} \in L_{t'}} p(\hat{t}|t')(v(t', \hat{t}) - b),$$
$$\Gamma_t(b) = p(t''|t)H_{t''}(b)(v(t, t'') - b) + \sum_{\hat{t} \in L_{t'}} p(\hat{t}|t)(v(t, \hat{t}) - b).$$

If (B.1) holds, then $\Gamma_t(b)$ is nonincreasing since

$$\begin{aligned}
\frac{d\Gamma_t(b)}{db} &= p(t''|t)h_{t''}(b)(v(t, t'') - b) - p(t''|t)H_{t''}(b) - p(L_{t'}|t) \\
&= p(t''|t)\left(h_{t''}(b)(v(t, t'') - b) - H_{t''}(b) - \frac{p(L_{t'}|t)}{p(t''|t)}\right) \\
&\leq p(t''|t)\left(h_{t''}(b)(v(t', t'') - b) - H_{t''}(b) - \frac{p(L_{t'}|t')}{p(t''|t')}\right) \\
&= \frac{p(t''|t)}{p(t''|t')}\big(p(t''|t')h_{t''}(b)(v(t', t'') - b) - p(t''|t')H_{t''}(b) - p(L_{t'}|t')\big) \\
&= \frac{p(t''|t)}{p(t''|t')}\frac{d\Gamma_{t'}(b)}{db} = 0,
\end{aligned}$$

where the last equality holds since $\Gamma_{t'}(b)$ is constant over $E_{t'}$.

S2.5 Proof of Lemma 5

The proof of Part (i) is standard and thus omitted. The proofs of the remaining statements are given in the order.

Proof of Part (ii). We only prove the statement for the case $m = 1$, since the argument is similar for the case $m = 0$. Suppose $E_{10} \cap E_{11} \neq \emptyset$ to the contrary. Consider any $b \in E_{10} \cap E_{11}$ and the expected payoff of each bidder i with $s_i = 1$ from bidding $b + \epsilon$ without learning s_j. With $\epsilon > 0$ being very small, the payoff from bidding $b + \epsilon$ will be close to $\Gamma_{1m'}$ when $s_j = m'$. So the expected payoff is close to $\alpha\Gamma_{11} + (1 - \alpha)\Gamma_{10}$, which is greater than $\alpha\Gamma_{11} + (1 - \alpha)\Gamma_{10} - k$ (i.e. the payoff from incurring the cost k and learning s_j), leading to a contradiction. ■

Proof of Part (iii). We first make a couple of observations before giving the proof of the statements. First, it is clear that $\Gamma_{00} = 0$ since $v_{00} = 0$. Second, if $\Gamma_1 > 0$, then we must also have $\Gamma_{11} > 0$, because $\Gamma_1 > 0$ implies that $\pi_1 < 1$ and $\bar{b}_1 < v_{11}$, and hence type-11 bidder can bid some $b \in (\bar{b}_1, v_{11})$ to guarantee at least $(1 - \pi_1)(v_{11} - b)$, a positive payoff.

We now prove that $\Gamma_t > 0$ for $t \neq 0, 00$. To see this, note that $\bar{b}_0 < v_{01}$ since the type $t = 0$ can only expect the ex-post value to be at most v_{01} and less than that sometimes. Then, the types $t = 10$ and 1 can bid some $b \in (\bar{b}_0, v_{10})$ to guarantee a positive payoff, which implies $\Gamma_{10}, \Gamma_1 > 0$. This in turn implies by the above observation that $\Gamma_{11} > 0$. If $\Gamma_{01} = 0$, then the expected payoff for bidder i with $s_i = 0$ who chooses to learn s_j would be $\alpha\Gamma_{00} + (1 - \alpha)\Gamma_{01} - k = -k < 0$, which is a contradiction. Thus, $\Gamma_{01} > 0$.

Lastly, to prove $\Gamma_0 = 0$, let $\tilde{\Omega} := \{t \in \Omega \backslash \{00\} \,|\, \underline{b}_t = \min_{t' \neq 00} \underline{b}'_t\}$. Then, it must be that $0 \in \tilde{\Omega}$, since otherwise some type $t \in \tilde{\Omega}$ with $t \neq 0, 00$ would have to earn zero equilibrium payoff, a contradiction. Also, no other type in $\tilde{\Omega}$ than type 0 can put a mass at $\underline{b}_0$ so the type $t = 0$ has to earn zero equilibrium payoff. ■

Proof of Part (iv). The statement is trivial if $t = m$ or $t = mm$ with $m = 0, 1$, since $t \in \Omega_t$. So consider type $t = mm'$ with $m \neq m'$ and let $\underline{b} := \min E^t$. If, to the contrary, there is some $b \in E_t \backslash E^t$, then either $b < \underline{b}$ or $b > \underline{b}$. The former case cannot happen since the winning probability, and thus payoff, from bidding such b is zero, contradicting Part (iii). In the latter case, b wins with a positive probability while there is no bid around b

belonging to E^t. So the type t can (slightly) reduce its bid from b without sacrificing its winning probability, a contradiction. ■

Proof of Part (v). Clearly, $\underline{b}_{00} = \overline{b}_{00} = v_{00}$. To show $\underline{b}_0 = v_{00}$, suppose $\underline{b}_0 = \min_{t \neq 00} \underline{b}_t > v_{00}$ to the contrary.[5] We consider two cases: (a) the type $t = 0$ puts a mass at $\underline{b}_0$; (b) $\underline{b}_0 < \underline{b}_t$ for all $t \neq 0, 00$ with no mass at $\underline{b}_0$. (Note that we can focus on these two cases since it is impossible to have $\underline{b}_t = \underline{b}_0$ for some $t \neq 0, 00$ without $t = 0$ placing a mass at $\underline{b}_0$.) In the former case, the type $t_i = 0$ would incur a loss by bidding $\underline{b}_0$ and tying with the rival's type $t_j = 0$. In the latter case, the equilibrium payoff of the type $t_i = 0$ bidding $\underline{b}_0 + \epsilon$ for small $\epsilon > 0$ is at most $\alpha(1 - \pi_0)H_0(\underline{b}_0 + \epsilon)(v_{00} - \underline{b}_0 - \epsilon) < 0$ (which realizes when facing the rival's type $t_j = 0$). ■

Proof of Part (vi). First, if $\underline{b}_{11} \leq \underline{b}_1$, then the type $t = 11$ would earn zero equilibrium payoff since $\Omega_{11} = \{1, 11\}$, a contradiction.

To prove $\underline{b}_1 > v_{00}$, we establish the following Claim first:

Claim S1. *There is some $b \in E_1 \cap E_{10}$ with $b > v_{00}$.*

Proof. Suppose not. We argue to draw a contradiction only for the case $\pi_0 > 0$, since a similar, in fact simpler, argument applies to the case $\pi_0 = 0$. Note first that $\overline{b}_1 > v_{00}$ and $\overline{b}_{10} > v_{00}$ since otherwise $t = 1$ or 10 would either earn zero equilibrium payoff or tie with $t = 0$ with a positive probability. Provided that the statement of this claim is not true, there must be some $b \in E_{10}$ such that for small $\epsilon > 0$, $(b - \epsilon, b) \subseteq E_1^c \cap E_{10}^c$. We then argue that (a) $(b - \epsilon, b) \subseteq E_{01}^c$ and (b) $(b - \epsilon, b) \subseteq E_0^c$. If (a) is not true, i.e. there is some $b' \in E_{01} \cap (b - \epsilon, b)$, then the type $t = 01$ with $\Omega_{01} = \{1, 10\}$ could profitably deviate to reduce its bid slightly below b' and enjoy the same winning probability. If (b) is not true, i.e. there is some $b'' \in E_0 \cap (b - \epsilon, b)$, then the type $t = 0$ with $\Omega_0 = \{0, 1, 10\}$ could slightly reduce its bid below b'' while enjoying the same winning probability against the type $t = 1$ and $t = 10$. While this can reduce the winning probability of $t = 0$ against the same type of rival, it is only beneficial since the value v_{00} from both bidders' type being $t = 0$ is smaller than b''. Given that (a) and (b) hold, the type $t = 10$ with $\Omega_{10} = \{0, 01\}$ can now reduce its bid below b and enjoy the same winning probability. ■

Suppose now that $\underline{b}_1 = v_{00}$ so $v_{00} \in E_1$ for a contradiction. First, $v_{00} \in E_1$ means $\Gamma_1 = \Gamma_1(v_{00}) = (1-\alpha)(1-\pi_0)(v_{10} - v_{00})H_0(v_{00})$.[6] If $t = 10$ bids arbitrarily close to v_{00}, its expected

[5]The fact that $\underline{b}_0 = \min_{t \neq 00} \underline{b}_t$ follows from the argument in Part (iv) above.

[6]Note that neither $t = 1$ nor $t = 01$ puts a mass at v_{00} in equilibrium.

payoff would be $(1-\pi_0)(v_{10}-v_{00})H_0(v_{00})$, which implies that $\Gamma_{10} \geq (1-\pi_0)(v_{10}-v_{00}) = \frac{\Gamma_1}{1-\alpha}$ or $\Gamma_1 \leq (1-\alpha)\Gamma_{10}$. Second, consider some $b \in E_1 \cap E_{10}$ with $b > v_{00}$. Suppose that $\pi_0 > 0$. (The argument for the case $\pi_0 = 0$ is similar and thus omitted.) Then, since $\Omega_{10} = \{0, 01\}$ and $\Omega_1 = \{0, 1, 01, 11\}$, we have $\Gamma_{10} = \Gamma_{10}(b) = [(1-\pi_0)H_0(b) + \pi_0 H_{01}(b)](v_{10} - b)$ and

$$\begin{aligned}\Gamma_1 = \Gamma_1(b) &= (1-\alpha)[(1-\pi_0)H_0(b) + \pi_0 H_{01}(b)](v_{10} - b) + \alpha[(1-\pi_1)H_1(b) + \pi_1 H_{11}(b)](v_{11} - b) \\ &= (1-\alpha)\Gamma_{10} + \alpha[(1-\pi_1)H_1(b) + \pi_1 H_{11}(b)](v_{11} - b).\end{aligned}$$

Thus,

$$\Gamma_1 - (1-\alpha)\Gamma_{10} = \alpha[(1-\pi_1)H_1(b) + \pi_1 H_{11}(b)](v_{11} - b) > 0,$$

since $v_{11} > b$ and $H_1(b) > 0$. This contradicts the above observation, $\Gamma_1 \leq (1-\alpha)\Gamma_{10}$. ■

Proof of Part (vii). Suppose to the contrary that $\bar{b}_0 = v_{00}$ so $E_0 = \{v_{00}\}$. To draw a contradiction, we consider two cases depending on whether $\pi_0 = 0$ or $\pi_0 > 0$. In the case $\pi_0 = 0$, the type $t = 10$ with $\Omega_{10} = \{0\}$ would like to bid as close to v_{00} as possible, which cannot be true in equilibrium. We thus suppose for a contradiction that $\pi_0 > 0$. Note first that $\underline{b}_{01} > v_{00}$ due to Parts (ii) and (v) of this lemma. Next, since $\Omega_{10} = \{0, 01\}$, we must have $E_{10} \subset E_{01}$.[7] Then, $v_{00} < \underline{b}_{01} \leq \underline{b}_{10}$. Also, $\underline{b}_1 < \underline{b}_{01}$ since, if $\underline{b}_{01} \leq \underline{b}_1$, the type $t = 01$ with $\Omega_{01} = \{1, 10\}$ could reduce its bid from $\underline{b}_{01}$ while enjoying the same winning probability. Given that $E_0 = \{v_{00}\}$, $\underline{b}_1 < \underline{b}_{01}$, and $\underline{b}_1 > v_{00}$ (by Part (v)), the type $t = 1$ with $\Omega_1 = \{0, 1, 01\}$ can reduce its bid below $\underline{b}_1$ while enjoying the same winning probability, a contradiction. ■

Proof of Part (viii). Let us first suppose to the contrary that $\underline{b}_{10} > v_{00}$, which, by Part (iv) of this lemma, means that $\min\{\underline{b}_1, \underline{b}_{10}\} > v_{00}$. Given this, neither $t = 0$ nor $t = 01$ has any incentive to place a bid in the interval $(v_{00}, \min\{\underline{b}_1, \underline{b}_{10}\})$ since it would be profitable to lower any bid belong to that interval, which means that no type places a bid in $(v_{00}, \min\{\underline{b}_1, \underline{b}_{10}\})$, contradicting with Part (i) of this lemma. So we must have $\underline{b}_{10} = v_{00}$. Also, if $\bar{b}_{10} < \underline{b}_1$, then there would be no bid placed in $(\bar{b}_{10}, \underline{b}_1)$ for the same reason as above, a contradiction.

We now show that it is not possible to have $\bar{b}_{10} > \underline{b}_1$, which will lead us to conclude $\bar{b}_{10} = \underline{b}_1$. First, we must have $\underline{b}_1 \in E_{10}$ since otherwise either $t = 0$ or $t = 01$ would

[7] If $E_{10} \not\subseteq E_{01}$, then the fact that $E_{10} \subseteq E_0 \cup E_{01}$ (due to Part (iii)) and $E_0 \cap E_{01} = E_{00} \cap E_{01} = \emptyset$ (due to Part (ii)) implies that the type $t = 10$ puts a mass at v_{00}, which cannot happen in equilibrium.

profitably reduce its bid below $\underline{b}_1$ so no type would place a bid slightly below $\underline{b}_1$. This means

$$\Gamma_1 = \Gamma_1(\underline{b}_1) = (1-\alpha)[(1-\pi_0)H_0(\underline{b}_1)+\pi_0 H_{01}(\underline{b}_1)](v_{10}-\underline{b}_1) = (1-\alpha)\Gamma_{10}(\underline{b}_1) = (1-\alpha)\Gamma_{10}. \quad \text{(S4)}$$

Second, it is clear that there must be some $b \in (\underline{b}_1, \bar{b}_{10})$ with $b \in E_1 \cap E_{10}$. For such b,

$$\begin{aligned}\Gamma_1 = \Gamma_1(b) &= (1-\alpha)[(1-\pi_0)H_0(b) + \pi_0 H_{01}(b)](v_{10} - b) + \alpha[(1-\pi_1)H_1(b) + \pi_1 H_{11}(b)](v_{11} - b) \\ &= (1-\alpha)\Gamma_{10} + \alpha[(1-\pi_1)H_1(b) + \pi_1 H_{11}(b)](v_{11} - b),\end{aligned}$$

which implies by (S4) that $\alpha[(1-\pi_1)H_1(b) + \pi_1 H_{11}(b)](v_{11} - b) = 0$, a contradiction

Lastly, if E_{10} is not a connected interval, then either $t = 0$ or type $t = 01$ could profitably reduce its bid somewhere in $E_{10}^c \cap [\underline{b}_{10}, \bar{b}_{10}]$. ■

Proof of Part (ix). With $\pi_1 \in (0,1)$, each bidder i with $s_i = 1$ must be indifference between learning and not learning s_j, so

$$\Gamma_1 = \alpha\Gamma_{11} + (1-\alpha)\Gamma_{10} - k. \quad \text{(S5)}$$

Note also that since $\bar{b}_{10} = \underline{b}_1 \in E_1$,we have

$$\Gamma_1 = \Gamma_1(\bar{b}_{10}) = (1-\alpha)\left[(1-\pi_0) + \pi_0 H_{01}(\bar{b}_{10})\right](v_{10} - \bar{b}_{10}) = (1-\alpha)\Gamma_{10}(\bar{b}_{10}) = (1-\alpha)\Gamma_{10}.$$

Substituting this into (S5) yields $k = \alpha\Gamma_{11}$. ■

S2.6 Proof of Lemma 6

Proof of Part (i). Since we already know $\bar{b}_{10} = \underline{b}_1$ from Part (viii) of Lemma 5, suppose for a contradiction that $\bar{b}_0 > \bar{b}_{10} = \underline{b}_1$. Then, there must be some interval $(b', b'') \subseteq (\bar{b}_{10}, \min\{\bar{b}_0, \underline{b}_{11}\})$ and $(b', b'') \subseteq E_0 \cap E_1$.[8] For any $b \in (b', b'')$,

$$\Gamma_1(b) = (1-\alpha)H_0(b)(v_{10} - b) + \alpha(1-\pi_1)H_1(b)(v_{11} - b), \quad \text{(S6)}$$

$$\Gamma_0(b) = \alpha H_0(b)(v_{00} - b) + (1-\alpha)[\pi_1 + (1-\pi_1)H_1(b)](v_{01} - b). \quad \text{(S7)}$$

[8]Recall from (vi) and $(viii)$ of Lemma 5 that $\bar{b}_{10} = \underline{b}_1 < \underline{b}_{11}$.

Multiply $1-\alpha$ and α to (S6) and (S7), respectively, and subtract the latter from the former to obtain

$$(1-\alpha)\Gamma_1(b) - \alpha\Gamma_0(b) = (1-\alpha)^2 H_0(b)v_{10} + \alpha(1-\alpha)[\pi_1(b - v_{01}) + (1-\pi_1)H_1(b)(v_{11} - v_{01})].$$

A contradiction arises since the LHS is constant on $E_0 \cap E_1$ whereas the RHS is increasing in b, which follows from the facts that $(1-\alpha)^2 v_{10} - \alpha^2 v_{00} = (1-\alpha)^2\beta > 0$ and $v_{11} - v_{01} = \beta > 0$.

It is straightforward that E_0 and E_{10} have to be connected intervals, since otherwise either $t = 0$ or $t = 10$ could profitably deviate by reducing its bid. ■

Proof of Part (ii). Clearly, we must have $\bar{b}_1 \geq \underline{b}_{11}$ since otherwise the type $t = 11$ can reduce its bid below $\bar{b}_1$ without decreasing the winning probability. The desired result will follow if we show that $\Gamma_{11}(b)$ is strictly increasing at any $b \in E_1$, since it means that the support of the bid distribution of type $t = 11$ must lie above that of type $t = 1$. To show it, note that at any $b \in E_1$

$$\Gamma_1 = \Gamma_1(b) = (1-\alpha)(v_{10} - b) + \alpha[\pi_1 H_1(b) + (1-\pi_1)H_{11}(b)](v_{11} - b) = (1-\alpha)(v_{10} - b) + \alpha\Gamma_{11}(b).$$

Then, $\Gamma_1(b)$ being constant implies that $\Gamma_{11}(b)$ is increasing in b, since the first term of the RHS of the last equality is decreasing in b. It is straightforward to show that E_1 and E_{11} are connected intervals. ■

S2.7 Proof of Lemma 7

Proof of Part (i). First, we claim that $E_0 \cup E_{01}$ must be a connected interval. That is, letting $\bar{b} = \max\{\bar{b}_0, \bar{b}_{01}\}$, $E_0 \cup E_{01} = [v_{00}, \bar{b}]$. Suppose to the contrary that there is some interval $(b, b') \subseteq [v_{00}, \bar{b}] \cap (E_0 \cup E_{01})^c$. This means that neither $t = 1$ nor $t = 10$ has an incentive to place a bid in (b, b') so no type places a bid in that interval, contradicting Part (i) of Lemma 5.

Now, the desired result will follow if we show that $\Gamma_{01}(b)$ is strictly increasing whenever $b \in E_0$ so $\Gamma_0(b)$ is constant, since it means that the support of $t = 01$ should lie above that of $t = 0$. To show this, note first

$$\begin{aligned}\Gamma_0(b) &= \alpha[(1-\pi_0)H_0(b) + \pi_0](v_{00} - b) + (1-\alpha)[(1-\pi_1)H_1(b) + \pi_1 H_{10}(b)](v_{01} - b) \\ &= \alpha[(1-\pi_0)H_0(b) + \pi_0](v_{00} - b) + (1-\alpha)\Gamma_{01}(b).\end{aligned}$$

From this, $\Gamma_0(b)$ being constant implies that $\Gamma_{01}(b)$ is increasing in b, since the first term of the RHS of the second equality is decreasing in b (for $b > v_{00}$). ■

Proof of Part (ii). First note that $t = 01$ or $t = 11$ must put no mass at any $b \in E_{01} \cup E_{11}$ in equilibrium, as can be shown in a straightforward manner, using the fact that Γ_{01} and Γ_{11} are both positive. We next show that $int(E_{01}) \cap int(E_{11}) = \emptyset$. If not, there is some interval $(b', b'') \subseteq E_{01} \cap E_{11}$ so for any $b \in (b', b'')$

$$\begin{aligned}\Gamma_{11}(b) &= [(1-\pi_1)H_1(b) + \pi_1 H_{11}(b)](v_{11} - b),\\ \Gamma_{01}(b) &= [(1-\pi_1)H_1(b) + \pi_1 H_{10}(b)](v_{01} - b).\end{aligned}$$

Note that $H_{10}(b) = 1$ since $\Omega_{01} = \{1, 10\}$ and $\overline{b}_{10} = \underline{b}_1 < \underline{b}_{11}$ by Parts (vi) and $(viii)$ of Lemma 5. Thus,

$$\Gamma_{11}(b) - \Gamma_{01}(b) = (1-\pi_1)H_1(b)(v_{11} - v_{01}) + \pi_1[H_{11}(b)(v_{01} - b) - v_{01} + b],$$

which cannot hold true since the LHS is constant while the RHS is increasing with b.[9]

We now show that $\Gamma_{11}(b)$ is strictly increasing at any $b \in E_1 \cap (int(E_{01}))^c$, which will imply (given the above finding, $int(E_{01}) \cap int(E_{11}) = \emptyset$) that the bid support of $t = 11$ lies above that of $t = 1$. To do so, one can write for any $b \in E_1 \cap (int(E_{01}))^c$

$$\begin{aligned}\Gamma_1 = \Gamma_1(b) =& (1-\alpha)[1 - \pi_0 + \pi_0 H_{01}(b)](v_{10} - b) + \alpha[\pi_1 H_1(b) + (1-\pi_1)H_{11}(b)](v_{11} - b)\\ =& (1-\alpha)[1 - \pi_0 + \pi_0 H_{01}(b)](v_{10} - b) + \alpha\Gamma_{11}(b). \qquad \text{(S8)}\end{aligned}$$

Since $H_{01}(b)$ is constant in $E_1 \cap (int(E_{01}))^c$, the first term in (S8) is decreasing with b so the second term, and thus $\Gamma_{11}(b)$, must be increasing. Given this, we must have $\overline{b}_1 \leq \underline{b}_{11}$ and thus $\overline{b}_1 = \underline{b}_{11}$ since if $\overline{b}_1 < \underline{b}_{11}$, then $t = 11$ could profitably reduce its bid below $\underline{b}_{11}$. ■

Proof of Part (iii). Suppose for a contradiction that $\overline{b}_0 > \underline{b}_1$. Then, there must be an interval (b', b'') such that $(b', b'') \subset (\underline{b}_1, \min\{\overline{b}_0, \underline{b}_{11}\})$ and $(b', b'') \subset E_0 \cap E_1$. The payoffs for type $t = 1$ and $t = 0$ from any $b \in (b', b'')$ are respectively

$$\begin{aligned}\Gamma_1(b) &= (1-\alpha)\left[\pi_0 H_{01}(b) + (1-\pi_0)H_0(b)\right](v_{10} - b) + \alpha(1-\pi_1)H_1(b)(v_{11} - b),\\ \Gamma_0(b) &= \alpha\left[\pi_0 + (1-\pi_0)H_0(b)\right](v_{00} - b) + (1-\alpha)\left[\pi_1 H_{10}(b) + (1-\pi_1)H_1(b)\right](v_{01} - b).\end{aligned}$$

[9]Differentiate the RHS with b to obtain $(1-\pi_1)h_1(b)(v_{11} - v_{01}) + \pi_1[h_{11}(b)(v_{01} - b) + 1 - H_{11}(b)] > 0$.

Then, we have $H_{01}(b) = 0$ and $H_{10}(b) = 1$ since $\overline{b}_{10} = \overline{b}_1 < b < \underline{b}_{01} = \overline{b}_0$ by Part (viii) of Lemma 5 and Part (i) of Lemma 7. We substitute these into the above payoffs and obtain

$$\begin{aligned}&(1-\alpha)\Gamma_1(b) - \alpha\Gamma_0(b)\\ =&(1-\alpha)^2(1-\pi_0)H_0(b)v_{10} + \alpha^2\pi_0 b + \alpha(1-\alpha)[\pi_1(b - v_{01}) + (1-\pi_1)H_1(b)(v_{11} - v_{01})],\end{aligned}$$

from which a contradiction arises since the LHS is constant over $(b', b'') \subset E_0 \cap E_1$ while the RHS is increasing in b. ■

Proof of Part (iv). If $\overline{b}_{01} < \underline{b}_1$, $t = 10$ could profitably reduce its bid below $\overline{b}_{10} = \underline{b}_1$. If $\overline{b}_{01} > \overline{b}_1$, $t = 01$ could profitably reduce its bid below $\overline{b}_{01}$. Also, if E_{01} is not a connected interval, either $t = 1$ or $t = 10$ could profitably reduce its bid somewhere in E_{01}^c. ■

Proof of Part (v). With $\pi_0 \in (0, 1)$, each bidder i with $s_i = 0$ must be indifferent between learning and not learning, which implies that

$$0 = \Gamma_0 = \alpha\Gamma_{00} + (1-\alpha)\Gamma_{01} - k = (1-\alpha)\Gamma_{01} - k,$$

and so $k = (1-\alpha)\Gamma_{01}$. ■

S2.8 Proof of Lemma 8

To prove this lemma and other results later, we first establish the following result:

Lemma S6. *Suppose that $\overline{b}_{10} = \overline{b}_{01}$ (that is, $H_{01}(\overline{b}_{10}) = 1$). If $k = \overline{k}_0$, then the solution π_1 of (7) coincides with that of (3) while π_0 defined in (8) is equal to zero. Moreover, if $k = \overline{k}_0$, then $\overline{b}_0 = \overline{b}_{01}$ in (9), which is in turn equal to $\overline{b}_0$ in (4).*

Proof. Note first that if $\overline{b}_{01} = \overline{b}_{10}$ or $H_{01}(\overline{b}_{10}) = 1$, then the equilibrium conditions produce the system of equations in (B.11) to (B.17) in the proof of Part (ii) of Proposition 6, whose solution is given by (7), (8), and (9). Recall that $\overline{k}_0$ defined by (5) satisfies

$$\overline{k}_0 = (1-\alpha)\pi_1(v_{01} - \overline{b}_0) = (1-\alpha)\pi_1\left(v_{01} - v_{11} + \frac{\overline{k}_0}{(1-\pi_1)\alpha} + \frac{\overline{k}_0}{1-\alpha}\right),$$

where the second equality follows from substituting $\overline{b}_0$ defined in (9). Rearranging this yields the same equation as (7) with $k = \overline{k}_0$, which means that if $k = \overline{k}_0$, the same π_1 solves both

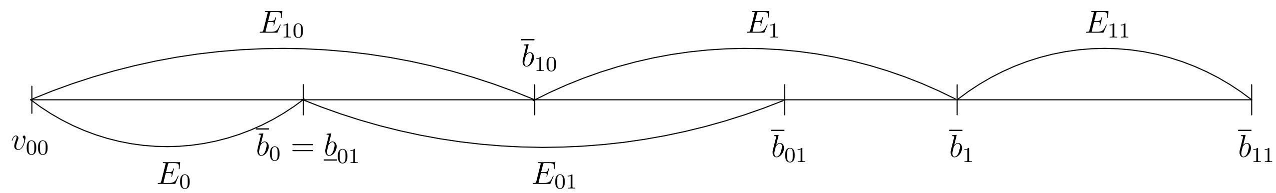

Figure 1: Bid supports for an alternative equilibrium with $\pi_1, \pi_0 > 0$

(3) and (7). Given this, by substituting (B.10), (5) can be written as

$$\bar{k}_0 = (1-\alpha)\pi_1(v_{01} - \bar{b}_0) = (1-\alpha)\left(v_{01} - \frac{(1-\alpha)\pi_1 v_{01}}{\alpha + (1-\alpha)\pi_1}\right) = \frac{v_{01}}{\frac{1}{(1-\alpha)\pi_1} + \frac{1}{\alpha}}, \tag{S9}$$

which means that π_0 given in (8) is equal to zero with $k = \bar{k}_0$.

Next, from (9), $\bar{b}_0 = \frac{k}{\alpha}$ and $\bar{b}_{01} = v_{01} - \frac{k}{(1-\alpha)\pi_1}$. Note that $\bar{b}_0$ is increasing in k and

$$\frac{\partial \bar{b}_0}{\partial k} = -\frac{\pi_1 - k\frac{\partial \pi_1}{\partial k}}{(1-\alpha)\pi_1^2} < 0,$$

where the inequality holds since π_1 is decreasing in k. Now, observe that

$$\bar{b}_{01} = v_{01} - \frac{\bar{k}_0}{(1-\alpha)\pi_1} = v_{01} - \frac{(1-\alpha)\pi_1(v_{01} - \bar{b}_0)}{(1-\alpha)\pi_1} = \bar{b}_0,$$

where the second equality follows from (5), and $\bar{b}_0$ in the RHS of the last equality is given by (4). Lastly, substituting $k = \bar{k}_0$ given by (S9) into $\bar{b}_0$ and $\bar{b}_{01}$, we have

$$\bar{b}_0 = \frac{v_{01}(1-\alpha)\pi_1}{\alpha + (1-\alpha)\pi_1} = \bar{b}_{01}.$$

■

To prove Lemma 8, suppose for contradiction that $k \geq \bar{k}_0$ but $\pi_0 > 0$. Note that by Lemma 5 and Lemma 7, the supports of the equilibrium bid distributions must look like those in Figure 1.

First, Part (*ix*) of Lemma 5 implies $k = (1-\alpha)\Gamma_{01}(\bar{b}_{10}) = (1-\alpha)(v_{01} - \bar{b}_{10})$, which yields after rearrangement

$$\bar{b}_{10} = v_{01} - \frac{k}{(1-\alpha)\pi_1}. \tag{S10}$$

Next, Part (v) of Lemma 7 implies

$$k = \alpha\Gamma_{11}(\bar{b}_1) = \alpha(1-\pi_1)(v_{11} - \bar{b}_1). \tag{S11}$$

Letting $z := (1-\pi_0) + \pi_0 H_{01}(\bar{b}_{10})$, the equilibrium payoff for type $t=1$ from bidding $\bar{b}_{10}$ is equal to

$$\Gamma_1(\bar{b}_{10}) = (1-\alpha)[(1-\pi_0)+\pi_0 H_{01}(\bar{b}_{10})](v_{10}-\bar{b}_{10}) = (1-\alpha)z\left(v_{10} - v_{01} + \frac{k}{(1-\alpha)\pi_1}\right), \tag{S12}$$

where the last equality follows from (S10). Also,

$$\Gamma_1(\bar{b}_1) = (1-\alpha)(v_{10} - \bar{b}_1) + \alpha(1-\pi_1)(v_{11} - \bar{b}_1) = (1-\alpha)(v_{10} - \bar{b}_1) + k, \tag{S13}$$

where the second equality follows from (S11). Then, we equate the RHS of (S12) and (S13) to obtain

$$\bar{b}_1 = v_{10} - z\left(v_{10} - v_{01} + \frac{k}{(1-\alpha)\pi_1}\right) + \frac{k}{1-\alpha}.$$

Also, from (S11), we have

$$\bar{b}_1 = v_{11} - \frac{k}{\alpha(1-\pi_1)}.$$

Equating these two expressions for $\bar{b}_1$, we obtain

$$\frac{v_{11} - v_{10} + z(v_{10} - v_{01})}{k} = \frac{1}{1-\alpha} + \frac{1}{\alpha(1-\pi_1)} - \frac{z}{(1-\alpha)\pi_1}. \tag{S14}$$

Now let $\pi_1(z)$ denote the solution of (S14) and $\pi_0(z)$ denote the corresponding equilibrium value of π_0. Clearly, $\pi_1(z)$ is decreasing in z since the LHS of (S14) is increasing in z while the RHS is decreasing in z and increasing in π_1. Note also that $\pi_1(1)$ is the solution of (7).

Claim S2. $\pi_0(z) = \frac{v_{01} - \frac{k}{(1-\alpha)\pi_1(z)} - \frac{k}{\alpha}}{v_{10} - \frac{k}{\alpha}}$ *while* $\pi_0(1)$ *is decreasing in* k*.*

Proof. Let us use Lemma 5 and Lemma 7 to write some equilibrium conditions as follows:

$$0 = \Gamma_0(\bar{b}_0) = \alpha(v_{00} - \bar{b}_0) + (1-\alpha)\pi_1(z)H_{10}(\bar{b}_0)(v_{01} - \bar{b}_0) \tag{S15}$$

$$\pi_1(z)H_{10}(\underline{b}_{01})(v_{01} - \underline{b}_{01}) = \Gamma_{01}(\underline{b}_{01}) = \Gamma_{01}(\bar{b}_{10}) = \pi_1(z)(v_{01} - \bar{b}_{10}) \tag{S16}$$

$$(v_{10} - \bar{b}_{10}) = \Gamma_{10}(\bar{b}_{10}) = \Gamma_{10}(\bar{b}_0) = (1-\pi_0(z))(v_{10} - \bar{b}_0) \tag{S17}$$

$$k = (1-\alpha)\Gamma_{01}(\bar{b}_{10}) = (1-\alpha)\pi_1(z)(v_{01} - \bar{b}_{10}). \tag{S18}$$

Rearranging (S15) and (S18) yields

$$\bar{b}_0 = \frac{\alpha v_{00} + (1-\alpha)\pi_1(z)H_{10}(\bar{b}_0)v_{01}}{\alpha + (1-\alpha)\pi_1(z)H_{10}(\bar{b}_0)} \text{ and } \bar{b}_{10} = v_{01} - \frac{k}{(1-\alpha)\pi_1(z)}, \tag{S19}$$

respectively. Note that

$$H_{10}(\bar{b}_0) = H_{10}(\underline{b}_{01}) = \frac{v_{01} - \bar{b}_{01}}{v_{01} - \bar{b}_0} = \frac{k}{(1-\alpha)\pi_1(z)(v_{01} - \bar{b}_0)}, \tag{S20}$$

where the first equality follows from the fact that $\bar{b}_0 = \underline{b}_{10}$, the second from (S16), and the third from substituting the expression for $\bar{b}_{01}$ in (S19). We can substitute (S20) into the expression for $\bar{b}_0$ in (S19) and solve for $\bar{b}_0$ to obtain $\bar{b}_0 = \frac{k}{\alpha}$. Next, rearranging (S17) yields $\pi_0(z) = \frac{\bar{b}_{10} - \bar{b}_0}{v_{10} - \bar{b}_0}$. Substituting $\bar{b}_{10} = v_{01} - \frac{k}{(1-\alpha)\pi_1(z)}$ and $\bar{b}_0 = \frac{k}{\alpha}$ into this equation yields the desired expression for $\pi_0(z)$.

The monotonicity of $\pi_0(1)$ with respect to k is established in Part (iii) of Proposition 6, where π_0 defined in (8) corresponds to $\pi_0(1)$. ■

Using the result in Claim S2, we obtain for any $k \geq k_0$

$$\pi_0(z) = \frac{v_{01} - \frac{k}{(1-\alpha)\pi_1(z)} - \frac{k}{\alpha}}{v_{10} - \frac{k}{\alpha}} \leq \frac{v_{01} - \frac{k}{(1-\alpha)\pi_1(1)} - \frac{k}{\alpha}}{v_{10} - \frac{k}{\alpha}} = \pi_0(1) \leq 0,$$

where the first inequality holds since $\pi_1(z) \leq \pi_1(1)$. The second inequality holds since $z = 1$ means $H_{01}(\bar{b}_{10}) = 1$ and thus $\pi_0(1) = 0$ at $k = \bar{k}_0$ by Lemma S6, while $\pi_0(1)$ decreases in k.

S2.9 Proof of Lemma 9

We prove $H_{01}(\bar{b}_{10}) = 1$ since it is equivalent to $\bar{b}_{01} = \bar{b}_{10}$.[10] Suppose for a contradiction that $H_{01}(\bar{b}_{10}) < 1$, which means that $z = (1 - \pi_0) + \pi_0 H_{01}(\bar{b}_{10}) < 1$. Given the supports of the equilibrium bid distributions as in Figure 1, we have

$$\begin{aligned} \Gamma_1(\bar{b}_{01}) &= (1-\alpha)(v_{10} - \bar{b}_{01}) + \alpha(1-\pi_1)H_1(\bar{b}_{01})(v_{11} - \bar{b}_{01}) \\ &= (1-\alpha)(v_{10} - \bar{b}_{01}) + \alpha(1-\pi_1)H_1(\bar{b}_{01})(v_{11} - \bar{b}_1 + \bar{b}_1 - \bar{b}_{01}) \\ &= (1-\alpha)(v_{10} - \bar{b}_{01}) + kH_1(\bar{b}_{01}) + \alpha(1-\pi_1)H_1(\bar{b}_{01})(v_{11} - \bar{b}_{01}), \end{aligned} \tag{S21}$$

[10] Note that there is no atom except at v_{00}.

where the last equality follows from (S11). Equating the RHS of (S21) with the RHS of (S13), we obtain

$$\bar{b}_1 - \frac{k(1-H_1(\bar{b}_{01}))}{1-\alpha+\alpha(1-H_1(\bar{b}_{01}))} = v_{11} - \frac{k}{\alpha(1-\pi_1)} - \frac{k(1-H_1(\bar{b}_{01}))}{1-\alpha+\alpha H_1(\bar{b}_{01})}. \tag{S22}$$

Also, $k = (1-\alpha)\Gamma_{01} = (1-\alpha)\Gamma_{01}(\bar{b}_{01}) = (1-\alpha)\left[\pi_1(z) + (1-\pi_1(z))H_1(\bar{b}_{01})\right]$, which yields

$$\bar{b}_{01} = v_{01} - \frac{k}{(1-\alpha)[\pi_1(z) + (1-\pi_1(z))H_1(\bar{b}_{01})]}.$$

Equating the RHS of this equation with that of (S22) and letting $\overline{H}_1 := H_1(\bar{b}_{01})$, we obtain

$$\frac{v_{11}-v_{01}}{k} = \frac{1-\overline{H}_1}{1-\alpha+\alpha(1-\pi_1(z))\overline{H}_1} + \frac{1}{\alpha(1-\pi_1(z))} - \frac{1}{(1-\alpha)[\pi_1(z)+(1-\pi_1(z))\overline{H}_1]}. \tag{S23}$$

Recall the definition of $\pi_1(z)$, i.e, the solution to (S14). Thus, $\pi_1(z)$ must solve (S14) and (S23) simultaneously. However, this cannot be true since

$$\begin{aligned}
\frac{v_{11}-v_{01}}{k} &= \frac{1-\overline{H}_1}{1-\alpha+\alpha(1-\pi_1(z))\overline{H}_1} + \frac{1}{\alpha(1-\pi_1(z))} - \frac{1}{(1-\alpha)[\pi_1(z)+(1-\pi_1(z))\overline{H}_1]} \\
&< \frac{1-\overline{H}_1}{1-\alpha+\alpha(1-\pi_1(1))\overline{H}_1} + \frac{1}{\alpha(1-\pi_1(1))} - \frac{1}{(1-\alpha)[\pi_1(1)+(1-\pi_1(1))\overline{H}_1]} \\
&< \frac{1-\overline{H}_1}{1-\alpha} + \frac{1}{\alpha(1-\pi_1(1))} - \frac{1}{(1-\alpha)[\pi_1(1)+(1-\pi_1(1))\overline{H}_1]} && \text{(S24)} \\
&< \frac{1}{1-\alpha} + \frac{1}{\alpha(1-\pi_1(1))} - \frac{1}{(1-\alpha)\pi_1(1)} = \frac{v_{11}-v_{01}}{k}, && \text{(S25)}
\end{aligned}$$

where the first inequality holds since the RHS of (S23) is strictly increasing in π_1, $\pi_1(z)$ is decreasing in z, and $z < 1$. The last equality follows from the fact that $\pi_1(1)$ solves (S14) with $z = 1$. The last inequality follows from the fact that the expression in (S24) is strictly decreasing in $\overline{H}_1$ whenever $\pi_1(1) > \frac{\sqrt{5}-1}{2}$. The proof is then completed by showing that if $k \le \bar{k}_0$, then $\pi_1(1) > \frac{\sqrt{5}-1}{2}$.

Claim S3. *If $k \le \bar{k}_0$, then $\pi_1(1)$, which solves (7) (or solves (S24) with $z = 1$), is greater than $\frac{\sqrt{5}-1}{2}$.*

Proof. We prove that if $\pi_1(1) \le \frac{\sqrt{5}-1}{2}$, then (a) the RHS of the equation in (S25) is no greater than 4 and (b) the LHS of the equation in (S25) is greater than 4, which means that the equation in (S25) cannot hold, a contradiction. To show (a) first, substitute $\pi_1(1) = \frac{\sqrt{5}-1}{2}$ into the RHS of the equation in (S25) to obtain $\frac{-2\sqrt{5}\alpha-2\alpha+\sqrt{5}+3}{2\alpha-2\alpha^2}$ and note that this expression is maximized and becomes equal to 4 at $\alpha = 1/2$, which implies that if $\pi_1(1) \leq \frac{\sqrt{5}-1}{2}$, then the RHS of the equation in (S25) is no greater than 4. To prove (b), we show that the statement holds for $k = \overline{k}_0$ so it holds for any $k < \overline{k}_0$ as well. Solve first (5) to obtain

$$\overline{k}_0 = \frac{(1-\alpha)\alpha\left(3-\alpha-2\beta+\alpha\beta-\sqrt{(\alpha\beta+\alpha-2\beta+1)^2+4\alpha(\beta-\alpha)}\right)}{2\left(\alpha^2-2\alpha+2\right)}.$$

Substitute this and $v_{11} - v_{01} = \beta$ to obtain

$$\frac{v_{11}-v_{01}}{\overline{k}_0} = \frac{1}{\alpha(1-\alpha)}\left[\frac{2\left(\alpha^2-2\alpha+2\right)\beta}{3-\alpha-2\beta+\alpha\beta-\sqrt{(\alpha\beta+\alpha-2\beta+1)^2+4\alpha(\beta-\alpha)}}\right] \geq 4,$$

where the inequality holds since $\alpha \in (1/2, 1)$ and so $\frac{1}{\alpha(1-\alpha)} > 4$, and the fact that the term in the square bracket is minimized and equal to 1 at $(\alpha, \beta) = (1, \frac{1}{2})$. ■

S2.10 Proof of Claim 1

Let us first obtain $H_1(\cdot)$, $H_0(\cdot)$, and $H_{10}(\cdot)$ by using the equilibrium conditions. For $b \in [\underline{b}_1, \overline{b}_1]$,

$$(1-\alpha)(v_{10}-\underline{b}_1) = \Gamma_1 = (1-\alpha)(v_{10}-b) + \alpha(1-\pi_1)H_1(b)(v_{11}-b),$$

so we have

$$H_1(b) = \frac{(1-\alpha)(b-\underline{b}_1)}{\alpha(1-\pi_1)(v_{11}-b)}. \tag{S26}$$

Next, for $b \in (\underline{b}_0, \overline{b}_0]$, $\Gamma_{10} = H_0(b)(v_{10}-b)$ so $H_0(b) = \frac{\Gamma_{10}}{v_{10}-b}$. Also,

$$0 = \Gamma_0 = \alpha H_0(b)(v_{00}-b) + (1-\alpha)\pi_1 H_{10}(b)(v_{01}-b).$$

We thus have

$$H_{10}(b) = \frac{\alpha H_0(b)(b-v_{00})}{(1-\alpha)\pi_1(v_{01}-b)} = \frac{\alpha(b-v_{00})\Gamma_{10}}{(1-\alpha)\pi_1(v_{01}-b)(v_{10}-b)},$$

where the second equality follows from substituting $H_0(b) = \frac{\Gamma_{10}}{v_{10}-b}$. Then, an optimal bid for $t_i = 01$ cannot be smaller than $\bar{b}_0$, since if bidding $b \in (\underline{b}_0, \bar{b}_0)$, $t_i = 01$ would obtain

$$\Gamma_{01}(b) = \pi_1 H_{10}(b)(v_{10} - b) = \frac{\alpha(b - v_{00})\Gamma_{10}}{(1-\alpha)(v_{01} - b)},$$

which is strictly increasing in $b \in (\underline{b}_0, \bar{b}_0) \subseteq [v_{00}, v_{01}]$. Thus,

$$\Gamma^*_{01}(k) = \max_{b \geq \bar{b}_0} \Gamma_{01}(b) = [\pi_1 + (1-\pi_1)H_1(b)](v_{01} - b) = \left[\pi_1 + \frac{(1-\alpha)(b - \underline{b}_1)}{\alpha(v_{11} - b)}\right](v_{01} - b), \quad \text{(S27)}$$

where the last equality follows from (S26). Let b^* denote a maximizer of (S27). If $b^* = \bar{b}_0$, then $\Gamma^*_{01}(k) = \pi_1(v_{01} - \bar{b}_0) = \frac{1-\alpha}{\alpha}(\bar{b}_0 - v_{00})$, where the second equality follows from rearranging (B.6). This payoff is decreasing in k since $\bar{b}_0$ is decreasing in k, as shown in the proof of Part (*iv*) of Proposition 5. So assume from now that $b^* > \bar{b}_0$—i.e., (S27) has an interior solution—, and apply the envelope theorem to (S27) to obtain

$$\frac{d\Gamma^*_{01}(k)}{dk} = \left[\frac{\partial \pi_1}{\partial k} - \frac{(1-\alpha)}{\alpha(v_{11} - b^*)}\left(\frac{\partial \underline{b}_1}{\partial k}\right)\right](v_{01} - b^*). \quad \text{(S28)}$$

To show that this expression is negative, let us first differentiate (B.6) with k to get

$$\begin{aligned} 0 &= -\alpha \frac{\partial \bar{b}_0}{\partial k} + (1-\alpha)\frac{\partial \pi_1}{\partial k}(v_{01} - \bar{b}_0) - (1-\alpha)\pi_1 \frac{\partial \bar{b}_0}{\partial k} \\ \Leftrightarrow \frac{\partial \pi_1}{\partial k} &= \frac{\alpha + (1-\alpha)\pi_1}{(1-\alpha)(v_{01} - \bar{b}_0)}\left(\frac{\partial \bar{b}_0}{\partial k}\right) = \frac{\alpha + (1-\alpha)\pi_1}{(1-\alpha)(v_{01} - \bar{b}_0)}\left(\frac{\partial \underline{b}_1}{\partial k}\right). \end{aligned}$$

This can be plugged into (S28) to yield after rearrangement

$$\frac{d\Gamma^*_{01}(k)}{dk} = \left[\frac{(\alpha + (1-\alpha)\pi_1)\alpha(v_{11} - b^*) - (1-\alpha)^2(v_{01} - \bar{b}_0)}{\alpha(v_{11} - b^*)}\right]\left(\frac{\partial \underline{b}_1}{\partial k}\right)(v_{01} - b^*) < 0,$$

where the inequality holds since $\frac{\partial \underline{b}_1}{\partial k} = \frac{\partial \bar{b}_0}{\partial k} < 0$, $b^* \in (v_{00}, v_{01})$, and

$$\begin{aligned} (\alpha + (1-\alpha)\pi_1)\alpha(v_{11} - b^*) - (1-\alpha)^2(v_{01} - \bar{b}_0) &> \alpha^2(v_{11} - v_{01}) - (1-\alpha)^2(v_{01} - v_{00}) \\ &= \alpha^2\beta - (1-\alpha)^2(1-\beta) > 0. \end{aligned}$$

S2.11 Proof of Claim 2

We show that the optimum for the maximization problem in (S27) is $b = \bar{b}_0$ if $k = \bar{k}_0$, which will imply that $(1-\alpha)\Gamma^*_{01}(\bar{k}_0) = (1-\alpha)\pi_1(v_{01} - \bar{b}_0) = \bar{k}_0$ by the definition of $\bar{k}_0$ given in (5). To show this, we first differentiate the maximand of (S27) to obtain for $b \in [\underline{b}_1, \bar{b}_1]$

$$\frac{d\Gamma_{01}(b)}{db} = -\pi_1 + \frac{1-\alpha}{\alpha}\left[1 - \frac{(v_{11} - \bar{b}_0)(v_{11} - v_{01})}{(v_{11} - b)^2}\right].$$

This is a decreasing function of b, meaning that $\Gamma_{01}(b)$ is concave over $[\underline{b}_1, \bar{b}_1]$. Thus, the optimum will be $\underline{b}_1 = \bar{b}_0$ if

$$\frac{d\Gamma_{01}(\bar{b}_0)}{db} = -\pi_1 + \frac{(1-\alpha)(v_{01} - \bar{b}_0)}{\alpha(v_{11} - \bar{b}_0)} \leq 0. \tag{S29}$$

To check this inequality, recall that $(v_{01} - \bar{b}_0) = \frac{\bar{k}_0}{(1-\alpha)\pi_1}$ from (5) and $v_{11} - b_0 = \frac{k}{(1-\pi_1)\alpha} + \frac{k}{1-\alpha}$ from Proposition 5. With $k = \bar{k}_0$,

$$\frac{(1-\alpha)(v_{01} - \bar{b}_0)}{\alpha(v_{11} - \bar{b}_0)} = \frac{(1-\alpha)\frac{\bar{k}_0}{(1-\alpha)\pi_1}}{\alpha\left(\frac{\bar{k}_0}{(1-\pi_1)\alpha} + \frac{\bar{k}_0}{1-\alpha}\right)} = \frac{(1-\alpha)(1-\pi_1)}{\pi_1(1-\alpha\pi_1)}$$

and thus

$$\frac{d\Gamma_{01}(\bar{b}_0)}{db} = -\pi_1 + \frac{(1-\alpha)(1-\pi_1)}{\pi_1(1-\alpha\pi_1)} = \frac{1 - \pi_1 - \pi_1^2 - \alpha(1 - \pi_1 - \pi_1^3)}{\pi_1(1-\alpha\pi_1)}.$$

So, $\frac{d\Gamma_{01}(\bar{b}_0)}{db} \leq 0$ if and only if $1 - \pi_1 - \pi_1^2 - \alpha(1 - \pi_1 - \pi_1^3) \leq 0$. Note first that

$$1 - \pi_1 - \pi_1^2 - \alpha(1 - \pi_1 - \pi_1^3) \leq 1 - \pi_1 - \pi_1^2 - \alpha(1 - \pi_1 - \pi_1^2).$$

Note also that if $1 - \pi_1 - \pi_1^2 \leq 0$, then we have $1 - \pi_1 - \pi_1^2 - \alpha(1 - \pi_1 - \pi_1^2) \leq 0$ since $\alpha \leq 1$. Thus, it suffices to show that $1 - \pi_1 - \pi_1^2 \leq 0$, which is equivalent to $\pi_1 \geq \frac{\sqrt{5}-1}{2}$. For this, observe that with $k = \bar{k}_0$, the solution of (3) coincides with that of (7) according to Lemma S6, and that the former is greater than $\frac{\sqrt{5}-1}{2}$ according to Claim S3.

S2.12 Proof of Claim 3

Using (7) and $v_{11} - v_{01} = \beta$, define

$$\mathcal{F}_1 := \frac{\beta}{k} - \frac{1}{1-\alpha} - \frac{1}{\alpha(1-\pi_1)} + \frac{1}{(1-\alpha)\pi_1}.$$

Since π_1 solves $\mathcal{F}_1 = 0$, we have that

$$\frac{\partial \pi_1}{\partial \alpha} = -\frac{\frac{\partial \mathcal{F}_1}{\partial \alpha}}{\frac{\partial \mathcal{F}_1}{\partial \pi_1}} = -\frac{\frac{1}{\alpha^2(1-\pi_1)} + \frac{1}{(1-\alpha)^2}\frac{1-\pi_1}{\pi_1}}{-\frac{1}{\alpha(1-\pi_1)^2} - \frac{1}{(1-\alpha)\pi_1^2}} = \frac{\pi_1(1-\pi_1)[(1-\alpha)^2\pi_1 + \alpha^2(1-\pi_1)^2]}{\alpha(1-\alpha)[(1-\alpha)\pi_1^2 + \alpha(1-\pi_1)^2]} > 0, \quad \text{(S30)}$$

$$\frac{\partial \pi_1}{\partial \beta} = -\frac{\frac{\partial \mathcal{F}_1}{\partial \beta}}{\frac{\partial \mathcal{F}_1}{\partial \pi_1}} = -\frac{\frac{1}{k}}{-\frac{1}{\alpha(1-\pi_1)^2} - \frac{1}{(1-\alpha)\pi_1^2}} = \frac{\alpha(1-\alpha)\pi_1^2(1-\pi_1)^2}{k[(1-\alpha)\pi_1^2 + \alpha(1-\pi_1)^2]} > 0. \quad \text{(S31)}$$

Next, using (8) and the facts that $v_{10} = \beta$ and $v_{01} = 1 - \beta$, define

$$\mathcal{F}_0 := \left(\beta - \frac{k}{\alpha}\right)\pi_0 - \left(1 - \beta - \frac{k}{(1-\alpha)\pi_1} - \frac{k}{\alpha}\right),$$

Since π_0 solves $\mathcal{F}_0 = 0$, we have that

$$\frac{\partial \pi_0}{\partial \alpha} = -\frac{\frac{\partial \mathcal{F}_0}{\partial \alpha}}{\frac{\partial \mathcal{F}_0}{\partial \pi_0}} = -\frac{\frac{k}{\alpha^2}\pi_0 + \frac{k}{(1-\alpha)^2\pi_1^2}\left(\pi_1 - (1-\alpha)\frac{\partial \pi_1}{\partial \alpha}\right)}{\beta - \frac{k}{\alpha}} < 0.$$

To see the inequality, note that the denominator is positive since $\beta - \frac{k}{\alpha} = v_{10} - \bar{b}_0 > 0$, and the second term in the numerator is

$$\begin{aligned}\frac{k}{(1-\alpha)\pi_1^2}\left(\frac{\pi_1}{1-\alpha} - \frac{\partial \pi_1}{\partial \alpha}\right) &= \frac{k}{(1-\alpha)\pi_1^2}\left(\frac{\pi_1}{1-\alpha} - \frac{\pi_1(1-\pi_1)[(1-\alpha)^2\pi_1 + \alpha^2(1-\pi_1)^2]}{\alpha(1-\alpha)[(1-\alpha)\pi_1^2 + \alpha(1-\pi_1)^2]}\right) \\ &= \frac{k}{(1-\alpha)\pi_1^2}\left(\frac{\pi_1^2[\alpha^2(1-\pi_1)^2 - (1-\alpha)(1-\alpha-\pi_1)]}{(1-\alpha)\alpha[(1-\alpha)\pi_1^2 + \alpha(1-\pi_1)^2]}\right) > 0,\end{aligned}$$

where the first equality follows from (S30) and the inequality holds since $\pi_1 > \frac{\sqrt{5}-1}{2} > \frac{1}{2}$ by Claim S3 in the proof of Lemma 9, and thus $(1-\alpha)(1-\alpha-\pi_1) < 0$ for $\alpha \in (\frac{1}{2}, 1)$. Similarly, we have that

$$\frac{\partial \pi_0}{\partial \beta} = -\frac{\frac{\partial \mathcal{F}_0}{\partial \beta}}{\frac{\partial \mathcal{F}_0}{\partial \pi_0}} = -\frac{\pi_0 + 1 - \frac{k}{(1-\alpha)\pi_1^2}\frac{\partial \pi_1}{\partial \beta}}{\beta - \frac{k}{\alpha}} = -\frac{\pi_0 + \frac{(1-\alpha)\pi_1^2}{(1-\alpha)\pi_1^2 + \alpha(1-\pi_1)^2}}{\beta - \frac{k}{\alpha}} < 0,$$

where the last equality follows from (S31).

S2.13 Proof of Claim 4

Suppose type $t = 0$ deviates to bid some $b \in E_{10} \backslash E_0 = E_{01}$. Then, $\Gamma_0(b) = p(0|0)(v_{00} - b) + p(1|0)H_{10}(b)(v_{10} - b)$. Observe that

$$\begin{aligned}\Gamma_0'(b) &= -p(0|0) + [p(10|0)h_{10}(b)(v_{10} - b) - p(10|0)H_{10}(b)] \\ &= -p(0|0) + \frac{p(10|0)}{p(10|01)}[p(10|01)h_{10}(b)(v_{10} - b) - p(10|01)H_{10}(b)] \\ &= -p(0|0) + \frac{p(10|0)}{p(10|01)}\Gamma_{01}'(b) = -p(0|0),\end{aligned}$$

where the penultimate equality holds since $\Gamma_{01}(b) = p(10|01)H_{10}(v_{10} - b)$ for $b \in E_{01}$.

Next, suppose type $t = 1$ deviates to bid some $b \in E_{10} = E_0 \cup E_{10}$. First, for $b \in E_0 \subset E_{10}$,

$$\Gamma_1(b) = p(0|1)H_0(b)(v_{10} - b) = \frac{p(0|1)}{p(0|10)}[p(0|10)H_0(b)(v_{10} - b)] = \frac{p(0|1)}{p(0|10)}\Gamma_{10}(b).$$

As the last term is constant in $b \in E_{10}$, so is $\Gamma_1(b)$. Second, for $b \in E_{01} = E_{10} \backslash E_0$,

$$\Gamma_1(b) = [p(0|1) + p(01|1)H_{01}(b)](v_{10} - b) = (1-\alpha)[p(0|10) + H_{01}(b)](v_{10} - b) = \frac{\Gamma_{10}(b)}{(1-\alpha)},$$

where the second equality holds since $p(0|1) = (1-\alpha)(1-\pi_0) = (1-\alpha)p(0|10)$ and $p(01|1) = (1-\alpha)\pi_0 = (1-\alpha)p(01|10)$. As the last terms is constant in $b \in E_{10}$, so is $\Gamma_1(b)$.

Lastly, suppose type $t = 01$ deviates to bid some $b \in E_0 = E_{10} \backslash E_{01}$. Note that

$$\Gamma_0(b) = \alpha[\pi_0 + (1-\pi_0)H_0(b)](v_{00} - b) + (1-\alpha)\pi_1 H_{10}(b)(v_{01} - b)$$

is constant across the interval E_0, which implies that the second term $H_{10}(b)(v_{01} - b)$ is increasing in $b \in E_0$ since the first term is decreasing. Thus, $\Gamma_{01}(b) = \pi_1 H_{10}(b)(v_{01} - b)$ is also increasing in $b \in E_0$.

S2.14 Proof of Claim 5

Note that since π_1 solve (3) for $k \in [\overline{k}_0, \overline{k}_1]$, we have from (3) that

$$\frac{\partial \pi_1}{\partial \alpha} = \frac{k(1-\pi_1)(\alpha + (1-\alpha)\pi_1)^2\left[(1-\alpha)^2 - \alpha^2(1-\pi_1)\right] + \pi_1(1-\pi_1)^2 v_{01}\alpha^2(1-\alpha)^2}{\alpha(1-\alpha)^2[k(\alpha + (1-\alpha)\pi_1)^2 + \alpha^2(1-\alpha)(1-\pi_1)^2 v_{01}]}. \tag{S32}$$

Next, rearrange (3) to obtain

$$\frac{v_{11}-v_{01}}{k}=\frac{1}{1-\alpha}+\frac{1}{(1-\pi_1)\alpha}-\frac{\alpha v_{01}}{k(\alpha+(1-\alpha)\pi_1)}$$
$$\Leftrightarrow v_{11}-\frac{k}{(1-\pi_1)\alpha}-\frac{k}{1-\alpha}=v_{01}-\frac{\alpha v_{01}}{\alpha+(1-\alpha)\pi_1}\ \Leftrightarrow\ \bar{b}_0=v_{01}-\frac{\alpha v_{01}}{\alpha+(1-\alpha)\pi_1}. \tag{S33}$$

We thus have

$$\begin{aligned}\frac{\partial\bar{b}_0}{\partial\alpha}&=-\frac{v_{01}\left(\pi_1-\alpha(1-\alpha)\frac{\partial\pi_1}{\partial\alpha}\right)}{(\alpha+(1-\alpha)\pi_1)^2}\\&=-\frac{v_{01}k[(1-\alpha)\pi_1+\alpha^2(1-\pi_1)^2-(1-\alpha)^2(1-\pi_1)]}{(1-\alpha)(\alpha+(1-\alpha)\pi_1)[k(\alpha+(1-\alpha)\pi_1)^2+\alpha^2(1-\alpha)(1-\pi_1)^2v_{01}]},\end{aligned} \tag{S34}$$

where the second equality follows from (S32). Using this,

$$\begin{aligned}&\frac{\partial(1-\alpha)(v_{10}-\bar{b}_0)}{\partial\alpha}\\&=-v_{10}+v_{01}-\frac{\alpha v_{01}}{\alpha+(1-\alpha)\pi_1}+\frac{v_{01}k\big((1-\alpha)\pi_1+\alpha^2(1-\pi_1)^2-(1-\alpha)^2(1-\pi_1)\big)}{(\alpha+(1-\alpha)\pi_1)\big(k(\alpha+(1-\alpha)\pi_1)^2+\alpha^2(1-\alpha)(1-\pi_1)^2v_{01}\big)}\\&=-\beta+\frac{1-\beta}{\alpha+(1-\alpha)\pi_1}\left[(1-\alpha)\pi_1+\frac{k\big((1-\alpha)\pi_1+\alpha^2(1-\pi_1)^2-(1-\alpha)^2(1-\pi_1)\big)}{k(\alpha+(1-\alpha)\pi_1)^2+\alpha^2(1-\alpha)(1-\pi_1)^2(1-\beta)}\right].\end{aligned}$$

It is straightforward to see that the last expression is decreasing in β and increasing in k. Thus, plugging $\beta=\frac{1}{2}$ and $k=\bar{k}_1=\alpha(1-\alpha)$ into that expression, we obtain after rearrangement

$$\begin{aligned}&\frac{\partial(1-\alpha)(v_{10}-\bar{b}_0)}{\partial\alpha}\\&<-\frac{1}{2}+\frac{1}{2(\alpha+(1-\alpha)\pi_1)}\left[(1-\alpha)\pi_1+\frac{2\big((1-\alpha)\pi_1+\alpha^2(1-\pi_1)^2-(1-\alpha)^2(1-\pi_1)\big)}{2(\alpha+(1-\alpha)\pi_1)^2+\alpha(1-\pi_1)^2}\right]\\&=\frac{-2+4\alpha-\alpha^2-2\alpha^3-(-4+6\alpha+4\alpha^2-4\alpha^3)\,\pi_1+(2-\alpha)(2\alpha-1)\pi_1^2}{2\,(\alpha+(1-\alpha)\pi_1)\,\big(2(\alpha+(1-\alpha)\pi_1)^2+\alpha(1-\pi_1)^2\big)}.\end{aligned} \tag{S35}$$

Letting $\kappa(\pi_1,\alpha)$ denote the numerator of (S35), it is a convex, quadratic function of π_1, which is thus maximized when π_1 is either 0 or 1. Since $\kappa(1,\alpha)=2-4\alpha\le 0$ and $\kappa(0,\alpha)=-2+4\alpha-\alpha^2-2\alpha^3\le 0$ for any $\alpha\in[1/2,1]$, the expression in (S35) is nonpositive, which means $\frac{\partial(1-\alpha)(v_{10}-\bar{b}_0)}{\partial\alpha}<0$, as desired.

S2.15 Proof of Claim 6

Consider any $k \in [\bar{k}_0, \bar{k}_1)$. First, note that $\bar{b}_0 = \frac{(1-\alpha)\pi_1 v_{01}}{\alpha+(1-\alpha)\pi_1}$ from rearranging (B.6), implying that $\bar{b}_0$ is decreasing in k since π_1 is decreasing in k. Second, since $\bar{b}_{10} = \bar{b}_0$, $\frac{\partial \bar{b}_{10}}{\partial \alpha}$ is given by (S34). Note that the expression within the square bracket of the numerator is

$$\begin{aligned}(1-\alpha)\pi_1 + \alpha^2(1-\pi)^2 - (1-\alpha)^2(1-\pi_1) &> (1-\alpha)\pi_1 + (1-\alpha)^2\left[(1-\pi_1)^2 - (1-\pi_1)\right] \\ &= (1-\alpha)\pi_1(1-(1-\alpha)(1-\pi_1)) > 0,\end{aligned}$$

and hence $\frac{\partial \bar{b}_{10}}{\partial \alpha} < 0$. Lastly, from the expression of $\bar{b}_0$ in (4) and the fact that π_1 is increasing in β by Part (ii) of Theorem 2, it is clear that $\bar{b}_0$ is decreasing in β, and so is $\bar{b}_{10}$.

Now consider any $k < \bar{k}_0$. Note that $\bar{b}_{10} = v_{01} - \frac{k}{(1-\alpha)\pi_1}$ is decreasing in k since π_1 is decreasing in k. Next,

$$\frac{\partial \bar{b}_{10}}{\partial \alpha} = -\frac{k[\pi_1 - (1-\alpha)\frac{\partial \pi_1}{\partial \alpha}]}{(1-\alpha)^2\pi_1^2} = -\frac{k\pi_1^2[\alpha^2\pi_1^2 - (2\alpha^2+\alpha-1)\pi_1 + 2\alpha - 1]}{(1-\alpha)^2\pi_1^2\alpha[(1-\alpha)\pi_1^2 + \alpha(1-\pi_1)^2]},$$

where the second equality follows from (S30). Observe that the terms in the square bracket in the numerator of the RHS of the last equality is a convex and quadratic function of π_1, so it is minimized when $\pi_1 = \frac{2\alpha^2+\alpha-1}{2\alpha^2}$. Thus,

$$\alpha^2\pi_1^2 - (2\alpha^2+\alpha-1)\pi_1 + 2\alpha - 1 \geq \frac{(1-\alpha)(2\alpha-1)(2\alpha^2+\alpha+1)}{4\alpha^2} \geq 0,$$

where the first inequality holds by substituting $\pi_1 = \frac{2\alpha^2+\alpha-1}{2\alpha^2}$ and after some rearrangement. This shows that $\frac{\partial \bar{b}_{10}}{\partial \alpha} < 0$. Lastly, note that $\bar{b}_{10} = \bar{b}_{01}$ and

$$\frac{\partial \bar{b}_{01}}{\partial \beta} = \frac{\partial}{\partial \beta}\left(v_{01} - \frac{k}{(1-\alpha)\pi_1}\right) = -1 + \frac{k}{(1-\alpha)\pi_1^2}\frac{\partial \pi_1}{\partial \beta} = -\frac{(1-\alpha)\pi_1^2}{(1-\alpha)\pi_1^2 + \alpha(1-\pi_1)^2} < 0,$$

where the last equality follows from substituting (S31).