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Welfare Lower Bounds and Strategyproofness in the Queueing Problem

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Welfare Lower Bounds and Strategyproofness in the Queueing Problem*

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Abstract

We investigate the implications of *welfare lower bounds* together with *queue-efficiency* and *strategyproofness* in the context of the queueing problem. As a consequence, we provide alternative characterizations of the k-pivotal mechanisms (Mitra and Mutuswami [13]). First, we introduce the *k-welfare lower bound,* which ensures that no agent is worse off than the case where she is assigned to the kth position in the queue without any monetary transfer. For each k, we show that the k-pivotal mechanisms generate the minimal budget deficit in each queueing problem among all mechanisms satisfying *queue-efficiency, strategyproofness* and the *k-welfare lower bound.* Next, we consider a well-known welfare lower bound, the *identical preferences lower bound* and show that when there are odd number of agents, the k-pivotal mechanisms with $k = \frac{n+1}{2}$ generate the minimal budget deficit in each queueing problem among all mechanisms satisfying *queue-efficiency, strategyproofness* and the *identical preferences lower bound.*

JEL Classification: C72, D63, D71, D82.

Keywords: Queueing problem, queue-efficiency, strategyproofness, k-pivotal mechanisms, k-welfare lower bound, identical preferences lower bound.

1 Introduction

A queueing problem concerns the following situation. A group of agents must be served in a facility (for example, a machine, a supercomputer or an expensive software in a university. See Maniquet [10], for more examples). The facility can handle only one agent at a time and agents incur waiting costs. We are interested in deciding the order to serve agents and the monetary transfers they should pay for the service. We assume that an agent's waiting cost is constant per unit of time, but that agents differ in their waiting costs. Furthermore, we assume that the utility of an agent is the amount of her monetary transfer minus her waiting costs. A mechanism assigns to each agent a position in the queue and a monetary transfer.

Queueing problems have been extensively analyzed both from a normative viewpoint (Chun [2], [3]; Maniquet [10]) as well as from a strategic viewpoint (Mitra [11]; Mitra and Mutuswami

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[13]). Also, several studies have analyzed the problem by combining the two approaches.[1] They are interested in mechanisms satisfying *queue-efficiency* and *strategyproofness* together with additional distributional requirements: *queue-efficiency* requires to minimize the aggregate waiting cost and *strategyproofness* induces each agent to report her waiting costs truthfully. The classic result of Holmström [8] implies in the context of queueing problems that a mechanism satisfies *queue-efficiency* and *strategyproofness* if and only if it is a VCG mechanism.[2] Imposing an additional distributional requirement gives us a subclass of VCG mechanisms.

Chun, Mitra and Mutuswami [4] additionally impose *egalitarian equivalence* (Pazner and Schmeidler [17]), which requires that there should be a reference bundle such that each agent enjoys the same utility between her consumption bundle and that reference bundle, and characterize a subfamily of VCG mechanisms. Also, a mechanism satisfying *no-envy* (Foley [6]) together with *queue-efficiency* and *strategyproofness* is characterized in Kayi and Ramaekers [9]. *No-envy* requires that no agent should end up with a higher utility by consuming what any other agent consumes. Our normative distributional requirements of welfare lower bounds provide a safety net to each agent. They play an important role in the fairness literature (Thomson [20]) and have been analyzed for several other economic problems[3]; however, in the queueing problem, they have not been investigated in depth.

From an agent's point of view, when there is no monetary transfer, the best queue position is obviously the first one and the worst is the last one. Consider a situation where all agents have equal rights over the use of the server and no agent is responsible for the existence of other agents. Then, each agent's "*stand-alone utility*" is the one where she is assigned the first queue position and no monetary transfer. A mechanism which guarantees each agent a welfare level at least as much as her stand-alone utility is said to meet the 1-*welfare lower bound.* Thus, the 1-*welfare lower bound* protects agents from the negative effects of circumstance for which they are not responsible, namely, the existence of other agents.

Since the total monetary transfer is not restricted in this problem, mechanisms that satisfy *queue-efficiency, strategyproofness* and the 1-*welfare lower bound* exist. However, if one requires that the mechanism does not generate a budget deficit in any queueing problem, then there is no *queue-efficient* and *strategyproof* mechanism meeting the 1-*welfare lower bound.* To obtain compatibility, we weaken the 1-*welfare lower bound* and introduce a family of welfare lower bounds, the k-*welfare lower bound* where $k \in \{1, 2, ..., n\}$. A mechanism that meets the k-*welfare lower bound* ensures each agent to enjoy a utility level at least as much as the one at which she is assigned to the kth position in the queue without any monetary transfer. That is, the k-*welfare lower bound* guarantees each agent a utility no smaller than her worst utility for the problem with k agents (see Subsection 3.1 for more details). Fortunately, *queue-efficiency, strategyproofness,* the k-*welfare lower bound* and *no-deficit* are compatible if $k \geq \frac{n+1}{2}$. As k increases, the lower bound on the utility decreases together with the minimum possible budget deficit compatible with the k-*welfare lower bound* (i.e., the maximum possible budget surplus compatible with the k-*welfare lower bound* increases). Hence, we face the usual trade-off between the level of fairness and the size of the budget surplus generated.

Our main result is the characterization, for each $k \in \{1, 2, ..., n\}$, of the parameterized family

[1] Such a combined analysis of strategic and fairness properties has also been done by Atlamaz and Yengin [1], Pápai [16], Mukherjee [15] and Yengin [22]- [26] in the larger class of problems of allocating heterogenous indivisible goods with monetary transfers.

[2] The family of VCG mechanisms is due to Vickrey [21], Clarke [5] and Groves [7].

[3] See, for instance, Atlamaz and Yengin [1], Moulin [14], Porter et al [18], and Yengin [24], [25], [26]. In the queueing problem, see Mitra [13] and Maniquet [10].

of mechanisms that generate the minimal deficit in each queueing problem among all *queue-efficient* and *strategyproof* mechanisms meeting the *k-welfare lower bound.* As it turns out, it is the family of k-pivotal mechanisms introduced and characterized by Mitra and Mutuswami [13]. For a given integer $k \in \{1, 2, ..., n\}$, a k-pivotal mechanism chooses an *efficient* queue and assigns the following transfers: An agent whose queue position is $r < k$ pays the sum of the waiting costs of all agents occupying queue positions $r + 1$ to k. An agent whose queue position is $s > k$ receives the sum of the waiting costs of all agents occupying queue positions k to $s-1$. The agent at the kth position in the queue pays and receives nothing. For $k = n$, we have the well-known "pivotal" mechanisms where each agent pays the sum of waiting costs of those served after him.

Mitra and Mutuswami [13] characterized the k-pivotal mechanisms with *queue-efficiency, equal treatment of equals, pairwise strategyproofness,* and *weak linearity.* They also showed that these mechanisms are *weakly group strategyproof.*[4] Here, we provide an alternative characterization of the k-pivotal mechanisms without relying on the technical axiom of *weak linearity.* Thus, our characterization of the k-pivotal mechanisms which uses only normative and strategic axioms provides an appealing justification for these mechanisms.

Next, we investigate one of the oldest fairness notion: guaranteeing each agent her utility from the equal split of the resources. Equal division is not well-defined in problems of allocating indivisible goods (such as queue positions). However, in such problems, an adaptation of the "equal-split lower bound," namely, the *identical preferences lower bound* (Moulin [14]), can be used. This bound requires that no agent is worse off than her utility at the Pareto-efficient and egalitarian allocation in the hypothetical economy where all agents have the same preferences as hers. Note that in classical fair division problems where equal division is well-defined, the equal-split lower-bound coincides with the *identical preferences lower bound.* Obviously, the *identical preferences lower bound* and the *k-welfare lower bound* are based on different fairness motivations. However, we show that in queueing problems, the two bounds are closely related. In particular, the *identical preferences lower bound* is equal to the expectation of k bounds. We also show that when there is an odd number of agents, the k-pivotal mechanisms with $k = \frac{n+1}{2}$ generate the minimal deficit in each queueing problem among all mechanisms satisfying *queue-efficiency, strategyproofness* and the *identical preferences lower bound.*

Given our characterizations together with the results in Mitra and Mutuswami [13], the k-pivotal mechanisms appear to be a prominent class of mechanisms satisfying several equity and strategic properties in queueing problems.

In what follows, the model and mechanisms are presented in Section 2. The results regarding *k-welfare lower bound*, the *identical preferences lower bound*, and budget properties are presented in Section 3. We conclude in Section 4.

2 The model

Let $N = \{1, \dots, n\}$, $n \geq 2$, be the set of agents. Each agent has one job to process and the server can process only one job at a time. Each job requires the same processing time, which without loss of generality, we normalize to one. A *queue* is an onto function $\sigma : N \to \{1, \dots, n\}$.

[4] *Pairwise strategy-proofness* requires that there does not exist a deviation for a coalition of size at most two making all deviating agents strictly better-off. *Weak group strategy-proofness* requires that there does not exist a deviation which makes all deviating agents strictly better-off. *Weak linearity* is a technical condition requiring that transfers vary in a linear fashion whenever an agent changes her announcement in a manner which does not change the efficient queue. *Equal treatment of equals* requires that agents with the same waiting costs should end up with the same utilities.

Agent i's position in the queue is denoted by σ_i. The *predecessors of i in the queue σ,* denoted by $P_i(\sigma)$, is the set $\{j \in N | \sigma_j < \sigma_i\}$. Similarly, the *followers of i in the queue σ,* denoted by $F_i(\sigma)$, is the set $\{j \in N | \sigma_j > \sigma_i\}$. The set of all possible queues is denoted by $\Sigma(N)$.

Each agent is identified with her *waiting cost per unit of time, $\theta_i \in \mathbb{R}_+$.*[5] A *queueing problem* is the profile of waiting costs of all agents $\theta = (\theta_i)_{i\in N}$. Let $\mathcal{Q}^N$ be the class of all problems for N. An *allocation* for $\theta \in \mathcal{Q}^N$ is a pair (σ, t), where $\sigma = (\sigma_i)_{i\in N}$ is the queue and $t = (t_i)_{i\in N}$ the vector of monetary transfer to agents. If agent i's queue position is σ_i, then she incurs a waiting cost of $(\sigma_i - 1)\theta_i$. An agent's net utility depends on her waiting cost and the transfer she receives. We assume that each agent $i \in N$ has a quasi-linear utility function, so that her utility from consuming the bundle (σ_i, t_i) is given by $u_i(\sigma_i, t_i; \theta_i) = -(\sigma_i - 1)\theta_i + t_i$. For each $\theta \in \mathcal{Q}^N$ and each $i \in N$, let $\theta_{-i} = (\theta_j)_{j\in N\setminus\{i\}}$ be the profile of waiting costs of all agents except agent i. For each $\theta \in \mathcal{Q}^N$ and each $k \in \{1, 2, ..., n\}$, let $\theta_{[k]}$ be the kth highest waiting cost in the problem θ.[6]

A queue σ is *efficient for the profile θ* if it minimizes the aggregate waiting cost of the agents, that is,

$$\sigma \in \underset{\sigma' \in \Sigma(N)}{\operatorname{argmin}} \sum_{i\in N} (\sigma'_i - 1)\theta_i.$$

The efficient queue is unique if $\theta_i \neq \theta_j$ for all $i, j \in N$, $i \neq j$. It is straightforward to check that a queue is efficient if and only if agents are served in the non-increasing order of their waiting costs. Let $E(\theta)$ be the set of all efficient queues for the profile θ.

A *mechanism* $\mu = (\sigma, t)$ associates with each problem $\theta \in \mathcal{Q}^N$, an allocation $\mu(\theta) = (\sigma, t) \in \Sigma(N) \times \mathbb{R}^n$. In this paper, we fix the set of agents and change the profile of waiting costs. To indicate the dependence on the problem θ, we denote the allocation as $\mu(\theta) = (\sigma(\theta), t(\theta))$. For each $i \in N$, $\mu_i(\theta) = (\sigma_i(\theta), t_i(\theta))$ represents agent i's position in the queue and her transfer.

For each $\theta \in \mathcal{Q}^N$, the total transfer $\sum_{i\in N} t_i(\theta)$ measures the budget deficit generated in the problem θ. If $\sum_{i\in N} t_i(\theta) > 0$, then the budget deficit equals $\sum_{i\in N} t_i(\theta)$. If $\sum_{i\in N} t_i(\theta) < 0$, then the budget surplus equals $-\sum_{i\in N} t_i(\theta)$. If $\sum_{i\in N} t_i(\theta) = 0$, then the budget is balanced.

2.1 The Mechanisms

We begin with two important properties of mechanisms. First, a mechanism should choose an efficient queue for each queueing problem.

Queue-efficiency: For each $\theta \in \mathcal{Q}^N$, $\sigma(\theta) \in E(\theta)$.

Remark 1. Our definition of a mechanism associates a unique queue to each queueing problem. Since $E(\theta)$ can contain more than one element, a *tie-breaking rule* is used to select a unique efficient queue whenever there is more than one such queue. We assume that there is an order of the agents according to which ties are broken. The same order is also used when we have to deal with subsets of the set of agents. Let $\mathcal{T}$ be the set of all possible tie-breaking rules for N and τ be a typical element of $\mathcal{T}$.

Our second property requires that an agent cannot strictly gain by misrepresenting her waiting cost, no matter what waiting costs other agents report.[7] Let $u_i(\mu_i(\theta'); \theta_i) = -(\sigma_i(\theta') -$

[5]Here, $\mathbb{R}_+$ denotes the non-negative orthant of the real line and $\mathbb{R}_{++}$ the positive orthant.

[6]All ties are taken into account in this order. For instance, if there are two agents whose waiting costs are the highest for θ, then $\theta_{[1]} = \theta_{[2]}$.

[7]See Thomson [20] for an extensive survey on strategyproofness.

$1)\theta_i + t_i(\theta')$ be the utility of agent i when the announced profile is θ' and her own waiting cost is θ_i.

Strategyproofness: For each $\theta \in \mathcal{Q}^N$, each $i \in N$ and each $\theta'_i \in \mathbb{R}_+$, $u_i(\mu_i(\theta);\theta_i) \geq u_i(\mu_i(\theta'_i,\theta_{-i});\theta_i)$.

We need the following notation. For each $\theta \in \mathcal{Q}^N$, suppose that there is an initial queue $\sigma(\theta)$ and agent i leaves the queue. The "induced" queue $\sigma(\theta_{-i})$ (of length $n-1$) for $N \setminus \{i\}$ is defined as follows:

$$\sigma_j(\theta_{-i}) = \begin{cases} \sigma_j(\theta) & \text{if } j \in P_i(\sigma(\theta)), \\ \sigma_j(\theta) - 1 & \text{if } j \in F_i(\sigma(\theta)). \end{cases} \tag{1}$$

In words, $\sigma(\theta_{-i})$ is the queue formed by removing agent i and moving all agents behind her up by one position. It is easy to see that $\sigma(\theta_{-i})$ is efficient for the profile θ_{-i} if $\sigma(\theta)$ is efficient for the profile θ.

A VCG mechanism chooses for each queueing problem an efficient queue and then, the transfer of each agent is determined in two parts. First, each agent pays the total waiting cost incurred by all other agents at the efficient queue chosen by the mechanism. Second, each agent receives an amount that only depends on the waiting costs of the other agents. For each $i \in N$, let h_i be a real-valued function such that for each $\theta \in \mathcal{Q}^N$, h_i depends only on θ_{-i}. Let $h = (h_i)_{i\in N}$ and $\mathcal{H}$ be the set of all h's.

The VCG mechanism associated with $h \in \mathcal{H}$ and $\tau \in \mathcal{T}$, $\mu^{h,\tau}$: Let $\mu^{h,\tau} \equiv (\sigma^\tau, t^{h,\tau})$ be such that for each $\theta \in \mathcal{Q}^N$, $\sigma^\tau(\theta) \in E(\theta)$, and each $i \in N$,

$$t_i^{h,\tau}(\theta) = -\sum_{j\in N\setminus\{i\}} (\sigma_j^\tau(\theta) - 1)\theta_j + h_i(\theta_{-i}). \tag{2}$$

Without loss of generality, we can write for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$h_i(\theta_{-i}) = \sum_{j\in N\setminus\{i\}} (\sigma_j^\tau(\theta_{-i}) - 1)\theta_j + g_i(\theta_{-i}) \tag{3}$$

where g_i is a real-valued function which depends only on θ_{-i}.

By substituting (3) in (2), together with (1), for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$t_i^{g,\tau}(\theta) = -\sum_{j\in F_i(\sigma^\tau(\theta))} \theta_j + g_i(\theta_{-i}). \tag{4}$$

Hence, for each $\tau \in \mathcal{T}$, $\mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ is a VCG mechanism if for each $\theta \in \mathcal{Q}^N$, $\sigma^\tau(\theta) \in E(\theta)$, and for each $i \in N$, the transfer is specified as in (4).

Remark 2. Holmström [8] showed that when preferences are quasi-linear and the domain of types is convex, a mechanism satisfies *queue-efficiency* and *strategyproofness* if and only if it is a VCG mechanism. In the queueing problem, since the type of each agent is her waiting cost, and moreover, the domain of waiting costs, $\mathbb{R}^n_+$, is convex, Holmström's result can be applied.

We are interested in a subclass of the VCG mechanisms, namely, the class of k-pivotal mechanisms introduced and characterized by Mitra and Mutuswami [13] on the basis of *group strategyproofness*. Let $k \in \{1, 2, ..., n\}$, $\tau \in \mathcal{T}$ and $P^{k,\tau} \equiv (\sigma^\tau, t^{k,\tau})$ be such that for each $\theta \in \mathcal{Q}^N$, $\sigma^\tau(\theta) \in E(\theta)$ and for each $i \in N$,

$$
t_i^{k,\tau}(\theta) = \begin{cases} -\sum_{j \in N: \sigma_i^\tau(\theta) < \sigma_j^\tau(\theta) \leq k} \theta_j & \text{if } \sigma_i^\tau(\theta) < k, \\ 0 & \text{if } \sigma_i^\tau(\theta) = k, \\ \sum_{j \in N: k \leq \sigma_j^\tau(\theta) < \sigma_i^\tau(\theta)} \theta_j & \text{if } \sigma_i^\tau(\theta) > k. \end{cases} \tag{5}
$$

For each $k \in \{1, 2, ..., n\}$, a k-pivotal mechanism assigns to each agent the following transfer: An agent whose queue position is $\sigma_i^\tau(\theta) = r < k$ pays the sum of the waiting costs of all agents occupying queue positions $r+1$ to k. The agent whose queue position is $\sigma_i^\tau(\theta) = k$ pays and receives nothing. An agent whose queue position is $\sigma_i^\tau(\theta) = s > k$ receives the sum of the waiting costs of all agents occupying queue positions k to $s-1$.

For our characterizations, we introduce an alternative specification of the transfers for the k-pivotal mechanism.

The k-pivotal mechanism associated with $\tau \in \mathcal{T}$, $\mathbf{P}^{k,\tau}$: Let $P^{k,\tau} = (\sigma^\tau, t^{k,\tau})$ be such that for each $\theta \in \mathcal{Q}^N$, $\sigma^\tau(\theta) \in E(\theta)$ and for each $i \in N$,

$$
t_i^{k,\tau}(\theta) = \begin{cases} -\sum\limits_{j \in F_i(\sigma^\tau(\theta))} \theta_j + \sum\limits_{l=k}^{n-1} (\theta_{-i})_{[l]} & \text{if } k \in \{1, 2, ..., n-1\}, \\ -\sum\limits_{j \in F_i(\sigma^\tau(\theta))} \theta_j & \text{if } k = n. \end{cases} \tag{6}
$$

For each $k \in \{1, 2, ..., n\}$, let $\mathcal{P}^k = \{P^{k,\tau}\}_{\tau \in \mathcal{T}}$ be the class of all k-pivotal mechanisms. By (6), for each $P^{k,\tau} = (\sigma^\tau, t^{k,\tau}) \in \mathcal{P}^k$, $P^{k,\tau} = \mu^{g,\tau}$ where for each $\theta \in \mathcal{Q}^N$ and each $i \in N$, if $k \in \{1, 2, ..., n-1\}$, then $g_i(\theta_{-i}) = \sum_{l=k}^{n-1} (\theta_{-i})_{[l]}$, and if $k = n$, then $g_i(\theta_{-i}) = 0$. For a k-pivotal mechanism, each agent first pays the sum of the waiting costs of her followers and then, she receives the amount $\sum_{l=k}^{n-1} (\theta_{-i})_{[l]} \geq 0$. Thus, for each $\theta \in \mathcal{Q}^N$ and each $i \in N$, $t_i^{k,\tau}(\theta) \geq -\sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j$.

Note that $P^{n,\tau} = \mu^{h,\tau}$ where for each $\theta \in \mathcal{Q}^N$ and each $i \in N$, $h_i(\theta_{-i}) = \sum_{j \in N \setminus \{i\}} (\sigma_j^\tau(\theta_{-i}) - 1)\theta_j$. From (2), for each $\theta \in \mathcal{Q}^N$, the transfer of each agent is equal to the externality she exerts. Thus, n-pivotal mechanisms are the well-known "pivotal" mechanisms.

It is easy to see that when $k = n$, the transfers specified in (6) are identical to the ones specified in (5). We also show that the transfers specified in (5) and (6) coincide for $k = 1, \ldots, n-1$.

Proposition 1. *Let $k \in \{1, 2, ..., n-1\}$ and $P^{k,\tau} = (\sigma^\tau, t^{k,\tau}) \in \mathcal{P}^k$. For each $\theta \in \mathcal{Q}^N$ and each $i \in N$,*

$$
-\sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j + \sum_{l=k}^{n-1} (\theta_{-i})_{[l]} = \begin{cases} -\sum_{j \in N: \sigma_i^\tau(\theta) < \sigma_j^\tau(\theta) \leq k} \theta_j & \textit{if } \sigma_i^\tau(\theta) < k, \\ 0 & \textit{if } \sigma_i^\tau(\theta) = k, \\ \sum_{j \in N: k \leq \sigma_j^\tau(\theta) < \sigma_i^\tau(\theta)} \theta_j & \textit{if } \sigma_i^\tau(\theta) > k. \end{cases} \tag{7}
$$

Proof: Let $k \in \{1, 2, ..., n-1\}$, $P^{k,\tau} = (\sigma^\tau, t^{k,\tau}) \in \mathcal{P}^k$, and $\theta \in \mathcal{Q}^N$. Note that for each $l \in \{1, 2, ..., n-1\}$ and each $i \in N$,

$$
(\theta_{-i})_{[l]} = \begin{cases} \theta_{[l+1]} & \text{if } \sigma_i^\tau(\theta) \leq l, \\ \theta_{[l]} & \text{if } \sigma_i^\tau(\theta) > l. \end{cases} \tag{8}
$$

Let $i \in N$. The proof is divided into 3 cases.

Case 1: $\sigma_i^\tau(\theta) = r < k$**.** Then, $\theta_i = \theta_{[r]}$ and $\sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j = \sum_{l=r}^{n-1} \theta_{[l+1]}$. Since $r < k$, for each $l \in \{k, ..., n-1\}$, $\sigma_i^\tau(\theta) < l$. By (8), $\sum_{l=k}^{n-1} (\theta_{-i})_{[l]} = \sum_{l=k}^{n-1} \theta_{[l+1]}$. Altogether,

$$t_i^{k,\tau}(\theta) = -\sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j + \sum_{l=k}^{n-1} (\theta_{-i})_{[l]} = -\sum_{l=r}^{n-1} \theta_{[l+1]} + \sum_{l=k}^{n-1} \theta_{[l+1]} = -\sum_{l=r}^{k-1} \theta_{[l+1]} = -\sum_{j \in N: r < \sigma_j^\tau(\theta) \le k} \theta_j, \tag{9}$$

as claimed.

Case 2: $\sigma_i^\tau(\theta) = k$**.** Then, $\theta_i = \theta_{[k]}$ and $\sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j = \sum_{l=k}^{n-1} \theta_{[l+1]}$. By (8), $\sum_{l=k}^{n-1} (\theta_{-i})_{[l]} = \sum_{l=k}^{n-1} \theta_{[l+1]}$. Altogether,

$$t_i^{k,\tau}(\theta) = -\sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j + \sum_{l=k}^{n-1} (\theta_{-i})_{[l]} = -\sum_{l=k}^{n-1} \theta_{[l+1]} + \sum_{l=k}^{n-1} \theta_{[l+1]} = 0, \tag{10}$$

as claimed.

Case 3: $\sigma_i^\tau(\theta) = s > k$**.** Then, $\theta_i = \theta_{[s]}$ and

$$\sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j = \begin{cases} \sum\limits_{l=s}^{n-1} \theta_{[l+1]} & \text{if } n > \sigma_i^\tau(\theta) > k, \\ 0 & \text{if } n = \sigma_i^\tau(\theta) > k. \end{cases} \tag{11}$$

By (8),

$$\sum_{l=k}^{n-1} (\theta_{-i})_{[l]} = \begin{cases} \sum\limits_{l=k}^{s-1} \theta_{[l]} + \sum\limits_{l=s}^{n-1} \theta_{[l+1]} & \text{if } n > \sigma_i^\tau(\theta) = s > k, \\ \sum\limits_{l=k}^{n-1} \theta_{[l]} & \text{if } n = \sigma_i^\tau(\theta) = s > k. \end{cases} \tag{12}$$

Altogether,

$$t_i^{k,\tau}(\theta) = -\sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j + \sum_{l=k}^{n-1} (\theta_{-i})_{[l]} = \sum_{l=k}^{s-1} \theta_{[l]} = \sum_{j \in N: k \le \sigma_j^\tau(\theta) < s} \theta_j, \tag{13}$$

as claimed. $\square$

3 Results

3.1 The k-Welfare Lower Bound

Imagine that there is only one agent in the society. Since she is the only agent, she would occupy the first position in the queue and incur no waiting cost. Hence, there is no need to give her a (positive or negative) monetary compensation. Her utility in this one-agent problem is called her *stand-alone utility.* Since no agent is responsible for the existence of other agents in the queueing problem, it may be fair to guarantee each agent her stand-alone utility. Our next axiom requires that each agent should be guaranteed her stand-alone utility.

Stand-alone lower bound: For each $\theta \in \mathcal{Q}^N$ and each $i \in N$, $u_i(\mu_i(\theta);\theta_i) \geq 0$.

A mechanism $\mu = (\sigma, t)$ meets the *stand-alone lower-bound* if for each $\theta \in \mathcal{Q}^N$ and each $i \in N$, $t_i(\theta) \geq (\sigma_i(\theta) - 1)\theta_i$, which implies that $\sum_{i\in N} t_i(\theta) \geq \sum_{i\in N}(\sigma_i(\theta)-1)\theta_i \geq 0$. Moreover, if $\theta_{[2]} > 0$, then $\sum_{i\in N} t_i(\theta) > 0$. Thus, if a mechanism meets the *stand-alone lower bound*, then it causes a budget deficit: the following axiom is violated.

No budget deficit: For each $\theta \in \mathcal{Q}^N$, $\sum_{i\in N} t_i(\theta) \leq 0$.

To satisfy *no budget deficit*, we need to weaken the *stand-alone lower-bound*. For $k \in \{1, 2, ..., n\}$, the *k-welfare lower bound* requires that no agent be worse off than if she is assigned to the kth position in the queue with a zero transfer. Since the *stand-alone lower bound* is the same thing as the 1-*welfare lower bound*, the *k-welfare lower bound* can be regarded as generalizing the idea of the *stand-alone lower bound* with $k \in \{1, 2, ..., n\}$. As k increases, the lower bound on the utility decreases. For each $k \in \{1, 2, ..., n-1\}$, if a mechanism satisfies the *k-welfare lower bound*, then it satisfies the $(k+1)$-*welfare lower bound*. Let $k \in \{1, 2, ..., n\}$.

k-welfare lower bound: For each $\theta \in \mathcal{Q}^N$ and each $i \in N$, $u_i(\mu_i(\theta);\theta_i) \geq -(k-1)\theta_i$.

A welfare lower bound in the fairness literature is typically determined by considering a particular "reference" allocation in a hypothetical problem and taking the welfare levels at this allocation as a benchmark for the utilities in the actual problem. Thus, an alternative motivation for the *k-welfare lower bound* can be given as follows. The 1-*welfare lower bound* guarantees each agent the utility obtained when she is the only agent in the problem. Now suppose that there are 2 agents. Without any transfer, the worst-case for an agent is if she is assigned to the last queue position. Taking this utility in the hypothetical 2-agent problem as the welfare lower bound in the actual problem leads to the 2-*welfare lower bound*. Similarly, for each $k \in \{1, 2, ..., n\}$, the *k-welfare lower bound* ensures that no agent is worse off than if she were assigned the worst position and a zero transfer in the hypothetical k-agent problem.

We are searching for the mechanisms that generate the minimal budget deficit in each queueing problem among the class of VCG mechanisms meeting the *k-welfare lower bound*. In general, there is no reason to expect the existence of a rule that dominates all others with respect to the budget deficit. As it turns out, for each k, the k-pivotal mechanism achieves our objective. Hence, our next characterization provides an alternative normative justification for these mechanisms.

Theorem 1. *Let $k \in \{1, 2, ..., n\}$. A mechanism minimizes the budget deficit in each queueing problem among all mechanisms satisfying queue-efficiency, strategyproofness and the k-welfare lower bound if and only if it is a k-pivotal mechanism.*

Proof: Let $k \in \{1, 2, ..., n\}$ and $P^{k,\tau} = (\sigma^\tau, t^{k,\tau}) \in \mathcal{P}^k$. By Remark 2, $P^{k,\tau}$ satisfies *queue-efficiency* and *strategyproofness*. By (7), for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$u_i(P_i^{k,\tau}(\theta);\theta_i) = \begin{cases} -(\sigma_i^\tau(\theta)-1)\theta_i - \sum\limits_{l=r+1}^{k} \theta_{[l]} & \text{if } \sigma_i^\tau(\theta) = r < k, \\ -(\sigma_i^\tau(\theta)-1)\theta_i & \text{if } \sigma_i^\tau(\theta) = k, \\ -(\sigma_i^\tau(\theta)-1)\theta_i + \sum\limits_{l=k}^{s-1} \theta_{[l]} & \text{if } \sigma_i^\tau(\theta) = s > k. \end{cases} \tag{14}$$

Let $i \in N$.

(i) If $\sigma_i^\tau(\theta) = r < k$, then for each $l \in \{r+1, r+2, ..., k\}$, $\theta_i \geq \theta_{[l]}$. By (14),

$$u_i(P_i^{k,\tau}(\theta);\theta_i) = -(\sigma_i^\tau(\theta)-1)\theta_i - \sum_{l=r+1}^{k} \theta_{[l]} \geq -(r-1)\theta_i - \sum_{l=r+1}^{k} \theta_i = -(k-1)\theta_i.$$

(ii) If $\sigma_i^\tau(\theta) = k$, then by (14), $u_i(P_i^{k,\tau}(\theta);\theta_i) = -(\sigma_i^\tau(\theta)-1)\theta_i = -(k-1)\theta_i$.

(iii) If $\sigma_i^\tau(\theta) = s > k$, then for each $l \in \{k, k+1, ..., s-1\}$, $\theta_i \leq \theta_{[l]}$. By (14),

$$u_i(P_i^{k,\tau}(\theta);\theta_i) = -(\sigma_i^\tau(\theta)-1)\theta_i + \sum_{l=k}^{s-1} \theta_{[l]} \geq -(s-1)\theta_i + \sum_{l=k}^{s-1} \theta_i = -(k-1)\theta_i.$$

Therefore, for each $\theta \in \mathcal{Q}^N$, each $i \in N$, and each possible queue position for agent i, $u_i(P_i^{k,\tau}(\theta);\theta_i) \geq -(k-1)\theta_i$, which implies that $P^{k,\tau}$ meets the *k-welfare lower bound.*

Conversely, let $\mu = (\sigma, t)$ be a mechanism satisfying *queue-efficiency, strategyproofness* and the *k-welfare lower bound.* By Remark 2, μ is a VCG mechanism, that is, there is a real-valued function $g : \mathbb{R}_+^{n-1} \to \mathbb{R}$ and $\tau \in \mathcal{T}$ such that $\mu = \mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ where $t^{g,\tau}$ is specified as in (4). By the *k-welfare lower bound*, for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$u_i(\mu_i^{g,\tau}(\theta);\theta_i) = -(\sigma_i^\tau(\theta)-1)\theta_i - \sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j + g_i(\theta_{-i}) \geq -(k-1)\theta_i,$$

which implies that

$$g_i(\theta_{-i}) \geq (\sigma_i^\tau(\theta) - k)\theta_i + \sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j. \tag{15}$$

The proof is divided into two cases:

Case 1: $k \in \{1, 2, ..., n-1\}$. Then, for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$g_i(\theta_{-i}) \geq \sum_{l=k}^{n-1} (\theta_{-i})_{[l]}. \tag{16}$$

Proof for Case 1: Suppose, by contradiction, that there is $\theta \in \mathcal{Q}^N$ and $i \in N$ such that

$$g_i(\theta_{-i}) < \sum_{l=k}^{n-1} (\theta_{-i})_{[l]}. \tag{17}$$

Let $\theta_i^* = (\theta_{-i})_{[k]}$ and $\theta^* = (\theta_i^*, \theta_{-i})$. Then, for each $l \in \{k, ..., n-1\}$, $(\theta^*)_{[l+1]} = (\theta_{-i})_{[l]}$. Therefore,[8]

$$\begin{aligned} (\sigma_i^\tau(\theta^*) - k)\theta_i^* + \sum_{j \in F_i(\sigma^\tau(\theta^*))} \theta_j^* &= (\theta_{-i})_{[k]} + (\theta_{-i})_{[k+1]} + \ldots + (\theta_{-i})_{[n-1]} \\ &= \sum_{l=k}^{n-1} (\theta_{-i})_{[l]}. \end{aligned} \tag{18}$$

[8] If $\sigma_i^\tau(\theta^*) < k$, then for all $i' \in N$ such that $\sigma_{i'} \in [\sigma_i^\tau(\theta^*), k]$, $\theta_{i'} = \theta_i^*$. Therefore, (18) holds. A similar observation can be made for the case when $\sigma_i^\tau(\theta^*) > k$.

Since $\theta^*_{-i} = \theta_{-i}$, by (17) and (18), $g_i(\theta^*_{-i}) < (\sigma^\tau_i(\theta^*) - k)\theta^*_i + \sum_{j \in F_i(\sigma^\tau(\theta^*))} \theta^*_j$, a contradiction to (15). Therefore, (16) holds. □

Case 2: $k = n$. Then, for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$g_i(\theta_{-i}) \geq 0. \tag{19}$$

Proof for Case 2: Suppose, by contradiction, that there is $\theta \in \mathcal{Q}^N$ and $i \in N$ such that

$$g_i(\theta_{-i}) < 0. \tag{20}$$

Let $\theta^*_i = (\theta_{-i})_{[n-1]}$ and $\theta^* = (\theta^*_i, \theta_{-i})$. Note that either $\sigma^\tau_i(\theta^*) = n$ or for each $j \in F_i(\sigma^\tau(\theta^*))$, we have $\theta^*_j = (\theta_{-i})_{[n-1]}$. Hence,

$$(\sigma^\tau_i(\theta^*) - n)\theta^*_i + \sum_{j \in F_i(\sigma^\tau(\theta^*))} \theta^*_j = (\sigma^\tau_i(\theta^*) - n)(\theta_{-i})_{[n-1]} + (n - \sigma^\tau_i(\theta^*))(\theta_{-i})_{[n-1]} = 0. \tag{21}$$

Since $\theta^*_{-i} = \theta_{-i}$, by (20) and (21), $g_i(\theta^*_{-i}) < (\sigma^\tau_i(\theta^*) - n)\theta^*_i + \sum_{j \in F_i(\sigma^\tau(\theta^*))} \theta^*_j$, a contradiction to (15). Therefore, (19) holds. □

For each $\theta \in \mathcal{Q}^N$, since

$$\begin{aligned}
\sum_{i \in N} t^{g,\tau}_i(\theta) &= -\sum_{i \in N} \sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j + \sum_{i \in N} g_i(\theta_{-i}) \\
&= -\sum_{l=2}^{n} (l-1)(\theta)_{[l]} + \sum_{i \in N} g_i(\theta_{-i}),
\end{aligned}$$

to minimize the budget deficit, we need to minimize $\sum_{i \in N} g_i(\theta_{-i})$. Therefore, for each $\theta \in \mathcal{Q}^N$ and each $i \in N$, (16) should hold as an equality for $k \in \{1, 2, ..., n-1\}$ and (19) for $k = n$. Therefore, by (4) and (6), we conclude that $\mu^{g,\tau}$ is a k-pivotal mechanism. ■

The next result follows from the proof of Theorem 1.

Corollary 1. *Let $k \in \{1, 2, ..., n\}$. A mechanism satisfies queue-efficiency, strategyproofness and the k-welfare lower bound if and only if it is a VCG mechanism $\mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ such that for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,*

$$\begin{aligned}
&g_i(\theta_{-i}) \geq \sum_{l=k}^{n-1} (\theta_{-i})_{[l]} && \text{if } k \in \{1, 2, ..., n-1\}, \text{ and} \\
&g_i(\theta_{-i}) \geq 0 && \text{if } k = n.
\end{aligned}$$

From Corollary 1, for each $k \in \{1, 2, ..., n\}$, if $\mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ is a VCG mechanism meeting the *k-welfare lower bound,* then for each $\theta \in \mathcal{Q}^N$ and each $i \in N$, $t^{g,\tau}_i(\theta) \geq t^{k,\tau}_i(\theta)$.

3.2 The Identical Preferences Lower Bound

For agent $i \in N$, consider a hypothetical "reference problem" $\theta^i \in \mathcal{Q}^N$ where all agents have preferences (in the queueing problem, waiting costs) identical to hers, that is, for each $j \in N$, $\theta^i_j = \theta_i$. Since all agents have equal rights and identical preferences, they should enjoy the same utility. The *identical preferences lower bound* (Moulin [14]) requires that each agent should be at least as well off as she would be, under *queue-efficiency, budget balance* and *equal treatment of equals,* if all other agents had preferences identical to hers. Note that if a mechanism satisfies *queue-efficiency, budget balance* and *equal treatment of equals,* then for each $i \in N$, in θ^i, each agent's utility is an equal share of $\sum_{j\in N} u_j(\mu_j(\theta^i);\theta^i_j) = -\sum_{j\in N}(\sigma_j(\theta^i)-1)\theta_i$, that is, $-\frac{n-1}{2}\theta_i$.

Identical preferences lower bound: For each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$u_i(\mu_i(\theta);\theta_i) \geq -\frac{n-1}{2}\theta_i.$$

Even though the *k-welfare lower bound* and the *identical preferences lower bound* are based on entirely different fairness considerations, they are closely related in the context of queueing problems. Suppose that agents do not know the size of the population. They expect to face one of the hypothetical problems with $k \in \{1,2,\dots,n\}$ agents. Remember that for each $k \in \{1,2,\dots,n\}$, the *k-welfare lower bound* guarantees each agent at least as much as her worst-case utility in the hypothetical k-agent problem without any transfer. One can take the expectation of these worst-case utilities of an agent over all hypothetical problems with k agents as a welfare lower bound. Call this bound as the "*expected k-welfare lower bound.*"

The *expected k-welfare lower bound* can also be thought of as a welfare lower bound from random arrival: assume that agents arrive randomly to join a queue, the server starts once all n agents arrive, and no monetary transfer is carried out. Hence, agent i who arrives in the kth position has a utility equal to $-(k-1)\theta_i$. To remove the unfairness associated with a particular arrival order, we can take the expectation of the utilities over all arrival orders and each agent would be guaranteed a utility at least as much as $-\frac{1}{n}\sum_{k=1}^{n}(k-1)\theta_i$.

A mechanism meets the *expected k-welfare lower bound* if for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$u_i(\mu_i(\theta);\theta_i) \geq -\frac{1}{n}\sum_{k=1}^{n}(k-1)\theta_i = -\frac{n-1}{2}\theta_i.$$

Therefore, the *expected k-welfare lower bound* coincides with the *identical preferences lower bound.*

Moreover, the *k-welfare lower bound* and the *identical preferences lower bound* are related in a more direct way. Since $-\frac{(n-1)}{2} = -(\frac{n+1}{2}-1)$, the *k-welfare lower bound* and the *identical preferences lower bound* coincide whenever $k = \frac{n+1}{2}$. Also, for each $k \leq \frac{n+1}{2}$, the *k-welfare lower bound* implies the *identical preferences lower bound,*[9] and the k-pivotal mechanisms meets the *identical preferences lower bound.* On the other hand, if $k > \frac{n+1}{2}$, then for each $\theta \in \mathcal{Q}^N$ and each $i \in N$ such that $\sigma^\tau_i(\theta) = k$, by (14), $u_i(P^{k,\tau}_i(\theta);\theta_i) = -(k-1)\theta_i < -\frac{(n-1)\theta_i}{2}$, implying that the k-pivotal mechanism with $k > \frac{n+1}{2}$ does not meet the *identical preferences lower bound.*

Hence, we have the following result.

Proposition 2. *Let $k \in \{1,2,...,n\}$. A k-pivotal mechanism meets the identical preferences lower bound if and only if $k \leq \frac{n+1}{2}$.*

[9] Note that this implication is true in general even without imposing *queue-efficiency* and *strategyproofness.*

In queueing problems, Mitra [12] showed the existence of a VCG mechanism meeting the *identical preferences lower bound* (together with *budget balance*). Here, we provide the full characterization of the class of VCG mechanisms meeting the *identical preferences lower bound* in queueing problems. For each $x \in \mathbb{R}_{++}$, let $\langle x \rangle_+$ denote the smallest integer greater than or equal to x and $\langle x \rangle_-$ denote the largest integer smaller than or equal to x.

Proposition 3. *(a) If a mechanism satisfies queue-efficiency, strategyproofness and the identical preferences lower bound, then it is a VCG mechanism $\mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ such that for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,*

$$g_i(\theta_{-i}) \geq \sum_{l=\left\langle \frac{n+1}{2} \right\rangle_+}^{n-1} (\theta_{-i})_{[l]}. \tag{22}$$

(b) If a VCG mechanism $\mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ is such that for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$g_i(\theta_{-i}) \geq \sum_{l=\left\langle \frac{n+1}{2} \right\rangle_-}^{n-1} (\theta_{-i})_{[l]}, \tag{23}$$

then it meets the identical preferences lower bound.

We omit the proof of Proposition 3, which is in line with the proof of Theorem 1.[10] Since the symmetrically balanced VCG mechanisms (Suijs [19]; Mitra [12]; Kayi and Raemaekers [9][11]) satisfy the three axioms of Proposition 3(a), they also satisfy (22).

If n is an odd number, then $\left\langle \frac{n+1}{2} \right\rangle_- = \left\langle \frac{n+1}{2} \right\rangle_+ = \frac{n+1}{2}$. Hence, for problems with odd number of agents, the following characterization follows from Theorem 1 and Proposition 3.

Corollary 2. *Let n be an odd number.*

(a) A mechanism satisfies queue-efficiency, strategyproofness and the identical preferences lower bound if and only if it is a VCG mechanism $\mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ such that for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$g_i(\theta_{-i}) \geq \sum_{l=\frac{n+1}{2}}^{n-1} (\theta_{-i})_{[l]}.$$

(b) A mechanism minimizes the budget deficit in each queueing problem among all mechanisms satisfying queue-efficiency, strategyproofness and the identical preferences lower bound if and only if it is a k-pivotal mechanism with $k = \frac{n+1}{2}$.

3.3 Budget Properties

It is well-known that the VCG mechanisms in general do not balance the budget. However, in queueing problems, the symmetrically balanced VCG mechanisms satisfy *budget balance*. On the other hand, the k-pivotal mechanisms do not satisfy *budget balance* except for the special case in which $n = 3$ and $k = \frac{n+1}{2}$ (see equality (26) and also Mitra and Mutuswami [13]). When $n = 3$, the 2-pivotal mechanisms coincide with the symmetrically balanced VCG mechanisms. The next result follows from Proposition 4 in Mitra [12].

[10] See the working paper version of our paper for the proof.
[11] Under the name of the largest equally distributed pairwise pivotal rule.

Proposition 4. *Let $n = 3$. A mechanism minimizes the budget deficit in each queueing problem among all VCG mechanisms meeting the 2-welfare lower bound if and only if it is a VCG mechanism that satisfies the identical preferences lower bound and budget balance.*

Since *budget balance* cannot be guaranteed, one can ask whether there are VCG mechanisms satisfying the k-*welfare lower bound* and *no budget deficit.* For $k = 1$, the answer is negative. When there are more than 3 agents, the negative result remains even if the 1-*welfare lower bound* is weakened to the 2-*welfare lower bound.* This negative result is generalized as follows.

Proposition 5. *If $k < \frac{n+1}{2}$, then no VCG mechanism satisfies the k-welfare lower bound and no budget deficit.*

Proof: Let $k \in \{1, 2, ..., n\}$, $P^{k,\tau} \equiv (\sigma^\tau, t^{k,\tau}) \in \mathcal{P}^k$, and $\mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ be a VCG mechanism satisfying the k-*welfare lower bound.* By Corollary 1, for each $\theta \in \mathcal{Q}^N$ and each $i \in N$, $t_i^{g,\tau}(\theta) \geq t_i^{k,\tau}(\theta)$, which implies that

$$\sum_{i \in N} t_i^{g,\tau}(\theta) \geq \sum_{i \in N} t_i^{k,\tau}(\theta). \tag{24}$$

Note that for each $\theta \in \mathcal{Q}^N$,

$$\begin{aligned} &-\sum_{i \in N} \sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j = -\sum_{l=2}^{n} (l-1)\theta_{[l]}, \text{ and} \\ &\sum_{i \in N} \sum_{l=k}^{n-1} (\theta_{-i})_{[l]} = \left((n-1) \sum_{l=k+1}^{n} \theta_{[l]} + (n-k)\theta_{[k]} \right) \text{ for } k = 1, \dots, n-1, \end{aligned} \tag{25}$$

where the second equality follows from (8). By (6) and (25), the budget deficit generated by $P^{k,\tau}$ is

$$\sum_{i \in N} t_i^{k,\tau}(\theta) = \begin{cases} \sum\limits_{l=1}^{n} (n-l)\theta_{[l]} & \text{if } k = 1, \quad (a) \\ -\sum\limits_{l=2}^{k} (l-1)\theta_{[l]} + \sum\limits_{l=k}^{n} (n-l)\theta_{[l]} & \text{if } k = 2, \dots, n. \quad (b) \end{cases} \tag{26}$$

If $k = 1$, by (26a), for each $\theta \in \mathbb{R}^n_{++}$, $\sum_{i \in N} t_i^{k,\tau}(\theta) > 0$ and by (24), $\sum_{i \in N} t_i^{g,\tau}(\theta) > 0$. If $2 \leq k < \frac{n+1}{2}$, let $\theta \in \mathcal{Q}^N$ be such that for each $i \in N$, $\theta_i = \theta' > 0$. By (24) and (26b), $\sum_{i \in N} t_i^{g,\tau}(\theta) \geq \sum_{i \in N} t_i^{k,\tau}(\theta) = \left(-\sum_{l=2}^{k} (l-1) + \sum_{l=k}^{n} (n-l) \right) \theta' = \frac{n(n-2k+1)}{2}\theta' > 0$. Hence, for $k < \frac{n+1}{2}$, both $P^{k,\tau}$ and $\mu^{g,\tau}$ violate *no budget deficit.* ■

Let $P^{k,\tau} \in \mathcal{P}^k$. By (26), if $k = 1$, then $P^{k,\tau}$ generates a non-negative budget deficit at all profiles and if $2 \leq k < \frac{n+1}{2}$, then $P^{k,\tau}$ does not necessarily generate a budget deficit: $\sum_{i \in N} t_i^{k,\tau}(\theta)$ may be positive, negative, or zero depending on θ. For instance, $\sum_{i \in N} t_i^{k,\tau}(\theta) \leq 0$ if for each $l \geq k$, $\theta_{[l]} = 0$. On the other hand, by Proposition 5.10 in Mitra and Mutuswami [13], if $k \geq \frac{n+1}{2}$, then $P^{k,\tau}$ generates a budget surplus for all profiles θ (i.e., $P^{k,\tau}$ satisfies *no budget deficit*). Hence, by Proposition 5, we have the following result.

Corollary 3. *There exists a VCG mechanism satisfying the k-welfare lower bound and no budget deficit if and only if $k \geq \frac{n+1}{2}$.*

Now we further weaken the requirement of *no budget deficit* by setting an upper bound on the budget deficit (or budget surplus). If the upper bound is equal to 0, then it is the requirement of *no budget deficit*. Let $T \in \mathbb{R}_+$ be given. Can we find VCG mechanisms meeting the *k-welfare lower bound* and generate a budget deficit or a budget surplus which never exceeds the given constant T? If this bound is satisfied, then the center knows that the largest amount of budget deficit or budget surplus it may face in any queueing problem, no matter what the waiting costs of the agents are, is the fixed amount T.

T-bounded budget deficit: For each $\theta \in \mathcal{Q}^N$, $\sum\limits_{i\in N} t_i(\theta) \leq T$.

T-bounded budget surplus: For each $\theta \in \mathcal{Q}^N$, $-\sum\limits_{i\in N} t_i(\theta) \leq T$.

Note that *T-bounded budget deficit* implies that the budget surplus is bounded below by $-T$ and *T-bounded budget surplus* implies that the budget deficit is bounded below by $-T$

On the domain of profiles $\mathbb{R}^n_+$, there is no upper bound on waiting costs. Hence, by (26), for $2 \leq k < \frac{n+1}{2}$, positive budget deficits or surpluses generated by mechanisms in $\mathcal{P}^k$ can be arbitrarily large. Therefore, for $2 \leq k < \frac{n+1}{2}$, the k-pivotal mechanisms satisfy neither *T-bounded budget deficit* nor *T-bounded budget surplus*.

On the other hand, by (26), for $k = 1$, the budget deficit $\sum_{i\in N} t_i^{k,\tau}(\theta)$ is not bounded above but it is bounded below by $\min_{\theta\in\mathcal{Q}^N} \sum_{i\in N} t_i^{k,\tau}(\theta) = \sum_{i\in N} t_i^{k,\tau}(\theta^0) = 0$ where θ^0 is such that for each $i \in N$, $\theta_i^0 = 0$ (i.e., the maximal budget surplus is 0). Thus, 1-pivotal mechanisms satisfy *T-bounded budget surplus* (with $T = 0$) but not *T-bounded budget deficit*. Similarly, for $k \geq \frac{n+1}{2}$, $\sum_{i\in N} t_i^{k,\tau}(\theta)$ is bounded above by $\max_{\theta\in\mathcal{Q}^N} \sum_{i\in N} t_i^{k,\tau}(\theta) = \sum_{i\in N} t_i^{k,\tau}(\theta^0) = 0$, which implies that for $k \geq \frac{n+1}{2}$, the k-pivotal mechanisms satisfy *T-bounded budget deficit* (with $T = 0$) but not *T-bounded budget surplus*.

Next we ask whether we can pin down the bounds on budget deficits and surpluses if we restrict ourselves to the class of problems in which the waiting costs are bounded above. For each $\overline{a} \in \mathbb{R}_+$, let $\mathcal{Q}^N(\overline{a}) = \{\theta \in \mathbb{R}^n_+ : \forall i \in N,\ 0 \leq \theta_i \leq \overline{a}\}$. Note that all our previous results hold on $\mathcal{Q}^N(\overline{a})$ for each $\overline{a} \in \mathbb{R}_+$. Moreover, we show that for any $k \in \{1, 2, ..., n\}$, the k-pivotal mechanisms satisfy *T-bounded budget deficit* as well as *T-bounded budget surplus* on this domain.

Proposition 6. *(a) If $k < \frac{n+1}{2}$, then for each $P^{k,\tau} \in \mathcal{P}^k$, each $\overline{a} \in \mathbb{R}_+$ and each $\theta \in \mathcal{Q}^N(\overline{a})$,*

$$-\frac{(k-1)(k-2)}{2}\overline{a} \leq \sum_{i\in N} t_i^{k,\tau}(\theta) \leq \frac{n(n-2k+1)}{2}\overline{a}. \tag{27}$$

(b) If $k \geq \frac{n+1}{2}$, then for each $P^{k,\tau} \in \mathcal{P}^k$, each $\overline{a} \in \mathbb{R}_+$ and each $\theta \in \mathcal{Q}^N(\overline{a})$,

$$-\frac{(k^2+k-2n)}{2}\overline{a} \leq \sum_{i\in N} t_i^{k,\tau}(\theta) \leq 0. \tag{28}$$

Proof: Let $k \in \{1, 2, ..., n\}$, $P^{k,\tau} \in \mathcal{P}^k$ and $\overline{a} \in \mathbb{R}_+$. Given $\theta \in \mathcal{Q}^N(\overline{a})$, let $\theta^k \in \mathcal{Q}^N(\overline{a})$ be such that for each $i \in N$, $\theta_i^k = \theta_{[k]}$. Let $\overline{\theta} \in \mathcal{Q}^N(\overline{a})$ be such that for each $i \in N$, $\overline{\theta}_i = \overline{a}$. Note that for each $l \in \{1, 2, ..., k\}$, $\theta_{[l]} \geq \theta_{[k]}$ and for each $l \in \{k, k+1, ..., n\}$, $\theta_{[l]} \leq \theta_{[k]}$. By (26), for each

$\theta \in \mathcal{Q}^N(\overline{a})$,

$$\begin{array}{l} \sum_{i \in N} t_i^{k,\tau}(\theta^k) \geq \sum_{i \in N} t_i^{k,\tau}(\theta), \\ \sum_{i \in N} t_i^{k,\tau}(\overline{\theta}) = \frac{n(n-2k+1)}{2}\overline{a} \text{ and} \\ \sum_{i \in N} t_i^{k,\tau}(\theta^k) = \frac{n(n-2k+1)}{2}\theta_{[k]}. \end{array} \tag{29}$$

Note that (26) can also be written as, for each $\theta \in \mathcal{Q}^N(\overline{a})$,

$$\sum_{i \in N} t_i^{k,\tau}(\theta) = \begin{cases} (n-2k+1)\theta_{[k]} + \sum_{l=k+1}^{n}(n-l)\theta_{[l]} & \text{if } k = 1, 2, \\ -\sum_{l=2}^{k-1}(l-1)\theta_{[l]} + (n-2k+1)\theta_{[k]} + \sum_{l=k+1}^{n}(n-l)\theta_{[l]} & \text{if } k = 3, \ldots, n. \end{cases} \tag{30}$$

(a) Suppose that $k < \frac{n+1}{2}$. Since $(n-2k+1) > 0$, by (29), for each $\theta \in \mathcal{Q}^N(\overline{a})$, $\sum_{i \in N} t_i^{k,\tau}(\overline{\theta}) \geq \sum_{i \in N} t_i^{k,\tau}(\theta^k) \geq \sum_{i \in N} t_i^{k,\tau}(\theta)$, which implies that $\max_{\theta \in \mathcal{Q}^N(\overline{a})} \sum_{i \in N} t_i^{k,\tau}(\theta) = \sum_{i \in N} t_i^{k,\tau}(\overline{\theta})$.

Let $\theta' \in \mathcal{Q}^N(\overline{a})$ be such that for each $l \in \{k, k+1, ..., n\}$, $\theta'_{[l]} = 0$ and if $k \geq 3$, then for each $l \in \{2, ..., k-1\}$, $\theta'_{[l]} = \overline{a}$. By (30), for $k \in \{1, 2\}$, $\sum_{i \in N} t_i^{k,\tau}(\theta') = 0$ and for $2 < k < \frac{n+1}{2}$, $\sum_{i \in N} t_i^{k,\tau}(\theta') = -\sum_{l=2}^{k-1}(l-1)\overline{a}$. That is, for each $k < \frac{n+1}{2}$, $\sum_{i \in N} t_i^{k,\tau}(\theta') = -\frac{(k-1)(k-2)}{2}\overline{a}$. Since $(n-2k+1) > 0$, by (30), for each $\theta \in \mathcal{Q}^N(\overline{a})$, $\sum_{i \in N} t_i^{k,\tau}(\theta') \leq \sum_{i \in N} t_i^{k,\tau}(\theta)$, which implies that $\min_{\theta \in \mathcal{Q}^N(\overline{a})} \sum_{i \in N} t_i^{k,\tau}(\theta) = \sum_{i \in N} t_i^{k,\tau}(\theta')$.

Altogether, for each $\theta \in \mathcal{Q}^N(\overline{a})$, (27) holds.

(b) Suppose that $k \geq \frac{n+1}{2}$. Let $\theta^0 \in \mathcal{Q}^N(\overline{a})$ be such that for each $i \in N$, $\theta_i^0 = 0$. Since $(n-2k+1) \leq 0$, by (29), for each $\theta \in \mathcal{Q}^N(\overline{a})$, $\sum_{i \in N} t_i^{k,\tau}(\theta) \leq \sum_{i \in N} t_i^{k,\tau}(\theta^k) \leq 0$, which implies that $\max_{\theta \in \mathcal{Q}^N(\overline{a})} \sum_{i \in N} t_i^{k,\tau}(\theta) = \sum_{i \in N} t_i^{k,\tau}(\theta^0) = 0$.

Let $\theta'' \in \mathcal{Q}^N(\overline{a})$ be such that for each $l \in \{2, 3, ..., k\}$, $\theta''_{[l]} = \overline{a}$ and if $k < n$, then for each $l \in \{k+1, ..., n\}$, $\theta''_{[l]} = 0$. By (26b), $\sum_{i \in N} t_i^{k,\tau}(\theta'') = (-\sum_{l=2}^{k}(l-1) + (n-k))\overline{a} = -\frac{(k^2+k-2n)}{2}\overline{a}$. Since $(n-2k+1) \leq 0$, by (30), for each $\theta \in \mathcal{Q}^N(\overline{a})$, $\sum_{i \in N} t_i^{k,\tau}(\theta'') \leq \sum_{i \in N} t_i^{k,\tau}(\theta)$, which implies that $\min_{\theta \in \mathcal{Q}^N(\overline{a})} \sum_{i \in N} t_i^{k,\tau}(\theta) = \sum_{i \in N} t_i^{k,\tau}(\theta'')$.

Altogether, for each $\theta \in \mathcal{Q}^N(\overline{a})$, (28) holds. ■

Note that by (24), (27), and (28), on $\mathcal{Q}^N(\overline{a})$, for each $k \in \{1, 2, ..., n\}$, if a VCG mechanism $\mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ meets the *k-welfare lower bound,* then it satisfies *T-bounded budget surplus*: for each $\theta \in \mathcal{Q}^N(\overline{a})$, the budget surplus $-\sum_{i \in N} t_i^{g,\tau}(\theta)$ is bounded above by $\frac{(k-1)(k-2)}{2}\overline{a} \geq 0$ if $k < \frac{n+1}{2}$, and by $\frac{(k^2+k-2n)}{2}\overline{a} \geq 0$ if $k \geq \frac{n+1}{2}$.

Finally, we point out the trade off between the level of welfare and the budget surplus generated by the k-pivotal mechanisms. As k increases, for each $k \in \{1, 2, ..., n-1\}$ and $\theta \in \mathcal{Q}^N$, the budget deficit and the total welfare decreases by

$$\sum_{i \in N} u_i(P_i^{k,\tau}(\theta); \theta_i) - \sum_{i \in N} u_i(P_i^{k+1,\tau}(\theta); \theta_i) = \sum_{i \in N} t_i^{k,\tau}(\theta) - \sum_{i \in N} t_i^{k+1,\tau}(\theta) = (n-k)\theta_{[k]} + k\theta_{[k+1]}. \tag{31}$$

4 Concluding Remarks

The *k-welfare lower bound* should not be confused with *k-fairness* introduced by Porter et al. [22] and characterized by Atlamaz and Yengin [1]. These authors considered the problem of assigning a single task among n agents where monetary transfers are possible. They defined the first best allocation as the Pareto-efficient (the task is assigned to an agent with the lowest cost and the budget is balanced) and egalitarian one. For each $k \in \{1, 2, ..., n\}$, the kth best allocation is the budget balanced and egalitarian allocation at which the task is assigned to an agent with the kth lowest cost. For each $k \in \{1, 2, ..., n\}$, *k-fairness* requires in each problem the utility at the kth best allocation to be a welfare lower bound.

One way to apply *k-fairness* to queueing problems may be the following: Define the first best allocation for $\theta \in \mathcal{Q}^N$ as the one where the queue is efficient, budget is balanced, and agents have equal utilities. Then, list all possible queues for θ in the non-decreasing order of aggregate waiting costs. Define the kth best allocation for θ as the one where the kth queue $\sigma^{[k]}(\theta)$ in this list of all possible queues is selected and utilities are equalized through budget balancing transfers. Note that $\sigma^{[1]}(\theta) \in E(\theta)$. For each $k \in \{1, 2, ..., n\}$, *k-fairness* defined in this way requires that for each $\theta \in \mathbb{R}^n_+$ and each $i \in N$,

$$u_i(\mu_i(\theta); \theta_i) \geq -\frac{\sum_{i \in N}(\sigma_i^{[k]}(\theta) - 1)\theta_i}{n}.$$

Clearly the *k-welfare lower bound, k-fairness,* and the *identical preferences lower bound* are conceptually different requirements with different underlying philosophical justifications. Note that the 1-*welfare lower bound* implies both *k-fairness* and the *identical preferences lower bound.* However, further logical relations may exist between these welfare bounds under *queue-efficiency* and *strategyproofness.* We have investigated the relationship between the *k-welfare lower bound* and the *identical preferences lower bound* in Section 3.2. In queueing problems, obviously, the 1-*welfare lower bound* and 1-*fairness* would coincide in the trivial scenario of $n = 1$. Investigating further relations between the *k-welfare lower bound* and *k-fairness* and between the *identical preferences lower bound* and *k-fairness* is an open question. Note that in the class of problems of allocating a single object (or multiple objects when valuation functions are additive) and money, under efficiency of assignment of objects and *strategyproofness,* the *identical preferences lower bound,* 1-*fairness,* and 2-*fairness* are equivalent (Remark 1 in Yengin [22], [23]).

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Appendix: Not for Publication

Proof of Proposition 3:

(a) Let $\mu = (\sigma, t)$ be a mechanism satisfying *queue-efficiency, strategyproofness* and the *identical preferences lower bound.* By Remark 2, $\mu = (\sigma, t)$ is a VCG mechanism, that is, there is a real-valued function $g : \mathbb{R}_+^{n-1} \to \mathbb{R}$ and $\tau \in \mathcal{T}$ such that $\mu = \mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ where $t^{g,\tau}$ is as in (4). By (4) and the *identical preferences lower bound*, for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$u_i(\mu_i^{g,\tau}(\theta); \theta_i) = -(\sigma_i^\tau(\theta) - 1)\theta_i - \sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j + g_i(\theta_{-i}) \geq -(\frac{n-1}{2})\theta_i,$$

which implies that $g_i(\theta_{-i}) \geq (\sigma_i^\tau(\theta) - \frac{n+1}{2})\theta_i + \sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j$. Since $\left\langle \frac{n+1}{2} \right\rangle_+ \geq \frac{n+1}{2}$, then

$$g_i(\theta_{-i}) \geq \left(\sigma_i^\tau(\theta) - \left\langle \frac{n+1}{2} \right\rangle_+ \right) \theta_i + \sum_{j \in F_i(\sigma^\tau(\theta))} \theta_j. \tag{32}$$

We need to show that for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$g_i(\theta_{-i}) \geq \sum_{l = \left\langle \frac{n+1}{2} \right\rangle_+}^{n-1} (\theta_{-i})_{[l]}. \tag{33}$$

Suppose, by contradiction, that there is $\theta \in \mathcal{Q}^N$ and $i \in N$ such that

$$g_i(\theta_{-i}) < \sum_{l = \left\langle \frac{n+1}{2} \right\rangle_+}^{n-1} (\theta_{-i})_{[l]}. \tag{34}$$

Let $\theta_i^* = (\theta_{-i})_{[k]}$ where $k = \left\langle \frac{n+1}{2} \right\rangle_+$ and $\theta^* = (\theta_i^*, \theta_{-i})$. For each $l \in \{k, ..., n-1\}$, $(\theta^*)_{[l+1]} = (\theta_{-i})_{[l]}$. Thus,[12] for $k = \left\langle \frac{n+1}{2} \right\rangle_+$,

$$\begin{aligned} (\sigma_i^\tau(\theta^*) - k)\theta_i^* + \sum_{j \in F_i(\sigma^\tau(\theta^*))} \theta_j^* &= (\theta_{-i})_{[k]} + (\theta_{-i})_{[k+1]} + \cdots + (\theta_{-i})_{[n-1]}, \\ &= \sum_{l = \left\langle \frac{n+1}{2} \right\rangle_+}^{n-1} (\theta_{-i})_{[l]}. \end{aligned} \tag{35}$$

Since $\theta_{-i}^* = \theta_{-i}$, by (34) and (35), $g_i(\theta_{-i}^*) < (\sigma_i^\tau(\theta^*) - \left\langle \frac{n+1}{2} \right\rangle_+)\theta_i^* + \sum_{j \in F_i(\sigma^\tau(\theta^*))} \theta_j^*$, a contradiction to (32). Hence, (33) holds.

[12]Even if $\sigma_i^\tau(\theta^*) \neq k$, the equation holds by using an argument similar to footnote 8.

(b) Let $\mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ be a VCG mechanism such that for each $\theta \in \mathcal{Q}^N$ and each $i \in N$, (23) holds. Then, by Corollary 1, $\mu^{g,\tau}$ satisfies the *k-welfare lower bound* where $k = \left\langle \frac{n+1}{2} \right\rangle_-$. Note that $\left\langle \frac{n+1}{2} \right\rangle_- \leq \frac{n+1}{2}$. Hence, by the *k-welfare lower bound* with $k = \left\langle \frac{n+1}{2} \right\rangle_-$, for each $\theta \in \mathcal{Q}^N$ and each $i \in N$, $u_i(\mu_i^{g,\tau}(\theta); \theta_i) \geq -(\left\langle \frac{n+1}{2} \right\rangle_- - 1)\theta_i \geq -(\frac{n+1}{2} - 1)\theta_i$, which implies that $\mu^{g,\tau}$ satisfies the *identical preferences lower bound*. ■

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