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# Demand Operators and the Dutta-Kar Rule for Minimum Cost Spanning Tree Problems

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**Abstract**

We investigate the implications of two demand operators, the *weak demand operator* and the *strong demand operator,* introduced by Granot and Huberman (1984) for *minimum cost spanning tree problems* (*mcstp*'s). The demand operator is intended to measure the maximum amount that each agent can ask to her followers in compensation for making a link to her. However, the original definition of the weak demand operator does not capture this idea and we propose its modification. Then, we introduce a procedure which enables us to calculate the maximum sequentially for each agent. By applying the modified weak demand operator to the irreducible *mcstp*'s, the Dutta-Kar allocation is obtained from any component-wise efficient initial allocation. For the strong demand operator, the Dutta-Kar allocation can be obtained if the procedure is initiated from any allocation in the irreducible core.

*JEL Classification*: C71.

*Keywords*: Minimum cost spanning tree problems, demand operators, irreducible matrix, Dutta-Kar rule, Prim algorithm.

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# 1 Introduction

A *minimum cost spanning tree problem* (*mcstp*) introduced by Claus and Kleitman (1973) is concerned with the construction of a minimal cost spanning tree (*mcst*) which provides for each agent a connection to the source, and the allocation of the total connection cost among the agents. Examples of the *mcstp* can be found easily: construction of communication networks such as telephone or cable television, construction of a drainage system, etc. An *mcst* can easily be found by using the Prim (1957) algorithm. On the other hand, for the fair allocation of total connection costs, many rules have been proposed: the Bird rule (Bird, 1976), the core and the nucleolus (Granot and Huberman, 1981, 1984), the Folk solution (Feltkamp et al., 1994; Bergantinos and Vidal-Puga, 2005, 2007; Lorenz and Lorenzo-Freire, 2009), the Kar rule (Kar, 2002), the Dutta-Kar rule (Dutta and Kar, 2004), the piecewise linear rules (Bogomolnaia and Moulin, 2010), the sequential equal contributions rules (Chun and Lee, 2012), and so on.

In this paper, we focus on the second question of fairly allocating the total cost among agents. In particular, we investigate the implication of two demand operators, the *weak demand operator* and the *strong demand operator,* introduced by Granot and Huberman (1984) for the *mcstp*. The demand operator is intended to measure the maximum amount that each agent can ask to her followers in compensation for making a link to her. For the weak demand operator, the amount that an agent can transfer to her immediate followers is determined by considering her opportunity cost. Since the original definition of the weak demand operator does not capture this idea properly, we propose its modification, the *modified weak demand operator*. For the strong demand operator, an agent is supposed to solve a maximization problem under the condition that no coalition of her followers together with any other agents has an incentive to make their own *mcst*.

We propose a procedure to apply the demand operators and calculate the resulting allocation as follows. First, a cost matrix is modified to its irreducible form before the operator is applied. The irreducible cost matrix is defined to be the cost matrix obtained by reducing the cost of each link as low as possible without affecting the minimum cost of the problem. Secondly, we rename agents using the order selected by the Prim algorithm when the *mcst* is constructed. Thirdly, we assume that the operating agent who is making a demand to her followers leaves after the operator is applied. The remaining agents cannot use the node of the leaving agent and an alternative *mcst* needs to be constructed. Since there may be many different ways to construct an *mcst* for the remaining agents, we impose a pre-specified selection rule for the construction of a new *mcst*. This rule selects as the new *mcst* one which keeps the original link structure as much as possible. Finally, we apply the operator sequentially to each agent from a given initial allocation.

As it turns out, our procedure provides another interesting justification for the Dutta-Kar rule. This rule is introduced by Dutta and Kar (2004) as a solution satisfying *cost monotonicity* and being a *core selection. Cost monotonicity* requires that the cost allocated to an agent does not increase if the cost of a link involving the agent decreases, nothing else changing. A rule is a *core selection* if no coalition of agents can be better off by building their own network. We show that if the modified weak demand operator is applied, then the Dutta-Kar allocation is obtained from any component-wise efficient initial allocation. For the strong demand operator, the Dutta-Kar allocation can be obtained if the procedure is initiated from any allocation in the irreducible core (that is, the core of the irreducible cost matrix).

This paper is organized as follows. In section 2, we introduce minimum cost spanning tree problems and cost allocation rules. In section 3, we present the two demand operators, the weak demand operator and the strong demand operator, and propose a modification of the weak demand operator. In section 4, we propose a procedure of applying the demand operators and show the coincidence results.

# 2 Preliminaries

## 2.1 The minimum cost spanning tree problem

Let $\mathbb{N} = \{1, 2, \cdots\}$ be a (finite or infinite) universe of all "potential" agents and $\mathcal{N}$ be the collection of non-empty, finite subsets of $\mathbb{N}$. A typical element of $\mathcal{N}$ is denoted by $N \equiv \{1, ..., n\}$ and $0$ is a special node called the *source*. We call each element of $N_0 \equiv N \cup \{0\}$ a node, and $\mathcal{N}_0 \equiv \{N_0 | N \in \mathcal{N}\}$.

Given $N_0 \in \mathcal{N}_0$, a *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ represents the cost of the direct link between any pair of nodes. For all $i, j \in N_0$, we assume that $c_{ij} \geq 0$ if $i \neq j$ and $c_{ij} = 0$ if $i = j$. Also, we assume that for all $i, j \in N_0$, $c_{ij} = c_{ji}$. The set of all cost matrices for $N_0$ is denoted by $\mathcal{C}_{N_0}$ and $\mathcal{C} \equiv \cup_{N_0 \in \mathcal{N}_0} \mathcal{C}_{N_0}$.

A *minimum cost spanning tree problem (mcstp)* is a pair $(N_0, C)$ where $N \in \mathcal{N}$ is a finite set of agents, $0$ is the source, and $C \in \mathcal{C}_{N_0}$ is the cost matrix.

A *graph* $g$ over $N_0$ is a subset of a complete graph $g_{N_0} \equiv \{(ij) | \forall i, j \in N_0, i \neq j\}$, whose element is an *arc*. A *path* from $i$ to $j$ in $g$ is a sequence of different arcs $\{(i_{k-1} i_k)\}_{k=1}^{K}$ that satisfies $(i_{k-1} i_k) \in g$ for all $k \in \{1, 2, \cdots, K\}$, $i_0 = i$ and

$i_K = j$. Two distinct nodes $i$ and $j \in N_0$ are *connected in* $g$ if there exists a path from $i$ to $j$, and a graph $g$ is *connected* if all pairs of nodes are connected in $g$. A *tree* is a connected graph with a unique path from any node to 0. We denote the set of all trees over $N_0$ as $\mathcal{T}_{N_0}$ and $\mathcal{T} \equiv \cup_{N_0 \in \mathcal{N}_0} \mathcal{T}_{N_0}$. For any $t \in \mathcal{T}_{N_0}$, let $t_{ij}$ be the unique path from $i$ to $j$ in $t$.

Given $t \in \mathcal{T}_{N_0}$ and $i, j \in N_0$, $i$ is a *predecessor* of $j$ in $t$ if there exist $k \in N_0$ such that $(ik) \in t_{0j}$. Let $P(j|t)$ be the set of all predecessors of $j$ in $t$. Agent $i$ is the *immediate predecessor* of $j$ in $t$ if $(ij) \in t_{0j}$. Let $p(j|t)$ be the immediate predecessor of $j$ in $t$. $j$ is a *follower* of $i$ in $t$ if $i \in P(j|t)$. Let $F(i|t)$ be the set of all followers of $i$ in $t$. Agent $j$ is an *immediate follower* of $i$ if $p(j|t) = i$. Let $f(i|t)$ be the set of all immediate followers of $i$ in $t$. For each $i \in f(0|t)$, $\{i\} \cup F(i|t)$ is a *component.*

For all $N_0 \in \mathcal{N}_0$ and all $C \in \mathcal{C}_{N_0}$, a *minimum cost spanning tree (mcst)* $t(N_0, C)$, is defined to be $\operatorname{argmin}_{t \in \mathcal{T}_{N_0}} \sum_{(ij) \in t} c_{ij}$. Let $m(N_0, C)$ be the minimum cost for the *mcstp* $(N_0, C)$, that is, $m(N_0, C) \equiv \sum_{(ij) \in t(N_0, C)} c_{ij}$.

Let $C_{|S_0}$ be the restriction of the cost matrix $C$ to the coalition $S_0 \subseteq N_0$. Bird (1976) associated an *mcstp* $(N_0, C)$ with a game $(N, c)$ where for each $S \subset N$, $c(S) = m(S_0, C_{|S_0})$. The *core* of the game $(N, c)$ is defined by

$$Core(N, c) \equiv \left\{ x \in \mathbb{R}^N \middle| \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S), \forall S \subset N \right\}.$$

Also, Bird (1976) introduced the *irreducible cost matrix* which is the cost matrix obtained by reducing the cost of each link as low as possible without affecting the total cost of the problem. For a given *mcstp* $(N_0, C)$ and its *mcst* $t(N_0, C)$, let $C^* = (c^*_{ij})_{i,j \in N_0}$ be the irreducible cost matrix of $C$, where for each $i, j \in N_0$, $c^*_{ij} = \max_{(kl) \in t_{ij}(N_0, C)} \{c_{kl}\}$. In addition, he proposed the irreducible core (denoted by $Core(N, c^*)$) as the core for the irreducible cost matrix, which is a subset of the core.

When there is no ambiguity, we use $P(i)$, $p(i)$, $F(i)$, $f(i)$, and $(S_0, C)$ instead of $P(i|t)$, $p(i|t)$, $F(i|t)$, $f(i|t)$, and $(S_0, C_{|S_0})$, respectively.

## 2.2 The Prim algorithm

Prim (1957) introduced an algorithm to find an *mcst*, now called the Prim algorithm, defined as follows.[1] For each $N_0 \in \mathcal{N}_0$ and each $C \in \mathcal{C}_{N_0}$,

[1] An alternative algorithm is proposed by Kruskal (1956).

Step 0: Let $A^0 \equiv \{0\}$ and $g^0 \equiv \emptyset$.

Step 1: Choose an ordered pair $(a^1 b^1)$ such that[2]

$$(a^1 b^1) = \underset{(ij)\in A^0\times(A^0)^c}{\operatorname{argmin}} c_{ij}.$$

Let $A^1 \equiv A^0 \cup \{b^1\}$ and $g^1 \equiv g^0 \cup \{(a^1 b^1)\}$.

Step k : Choose an ordered pair $(a^k b^k)$ such that

$$(a^k b^k) = \underset{(ij)\in A^{k-1}\times(A^{k-1})^c}{\operatorname{argmin}} c_{ij}.$$

Let $A^k \equiv A^{k-1} \cup \{b^k\}$ and $g^k \equiv g^{k-1} \cup \{(a^k b^k)\}$.

The algorithm terminates at step $n$ and $g^n$ is the *mcst* for $(N_0, C)$.

From now on, for a given *mcstp*, we rename each agent by the step chosen in the Prim algorithm, i.e., agent 1 is chosen in the first step of the Prim algorithm and agent $i$ in the $i^{th}$ step. When several agents have the same cost at some step, the Prim algorithm arbitrarily chooses one agent. We will avoid this difficulty by restricting the domain of permissible cost matrices.

$$\begin{aligned}\tilde{\mathcal{C}}_{N_0} \equiv &\{C \in \mathcal{C}_{N_0} | C \text{ induces a unique agent in each step of the Prim algorithm}\},\\ \tilde{\mathcal{C}} \equiv &\bigcup_{N_0\in\mathcal{N}_0} \tilde{\mathcal{C}}_{N_0}.\end{aligned}$$

An *mcst* may not be unique on $\tilde{\mathcal{C}}$. For example, let $N = \{1, 2, 3\}$ and $c_{01} = 6$, $c_{12} = 2$, $c_{13} = c_{23} = 3$ and let all other costs be larger than 6. Even though the *mcst* is not unique, we note that a unique agent is selected at each step of the Prim algorithm.

## 2.3 Rules

For each $N \in \mathcal{N}$, a *cost allocation* is a vector $y = (y_i)_{i\in N} \in \mathbb{R}^N_+$, where for each $i \in N$, $y_i$ is the *cost assigned to agent* $i$. Let $\mathcal{Y}_N$ be the set of all cost allocations for $N$. For each $N_0 \in \mathcal{N}_0$, a *cost allocation rule*, or a *rule*, is a

[2] Given $A \subseteq N_0$, $A^c \equiv N_0 \setminus A$.

function $\varphi : \mathcal{C}_{N_0} \to \mathcal{Y}_N$, which associates to each $N_0 \in \mathcal{N}_0$ and each $C \in \mathcal{C}_{N_0}$ a cost allocation $\varphi(N_0, C) \equiv (\varphi_i(N_0, C))_{i\in N}$.

Now we introduce two rules for the problem, the Bird rule and the Dutta-Kar rule (DK rule). For the Bird rule, each agent is sequentially connected to the source by the Prim algorithm and pays the additional cost incurred by her inclusion in the *mcst*.

**Bird rule**, $\varphi^B$: For all $N_0 \in \mathcal{N}_0$, all $C \in \tilde{\mathcal{C}}_{N_0}$, and all $i \in N$, $\varphi_i^B(N_0, C) = c_{p(i)i}$.

A rule is a *core selection* if no coalition of agents can be better off by building their own network. On the other hand, *cost monotonicity* requires that the cost allocated to agent $i$ does not increase if the cost of a link involving agent $i$ goes down, nothing else changing. The Bird rule is a *core selection* but fails to satisfy *cost monotonicity*.

Dutta and Kar (2004) proposed another rule for the problem, the DK rule $\varphi^{DK}$, which is also a *core selection* and satisfies *cost monotonicity*. This rule is defined as follows: for all $N_0 \in \mathcal{N}_0$ and all $C \in \tilde{\mathcal{C}}_{N_0}$,

Step 0: Let $A^0 \equiv \{0\}$, $g^0 \equiv \emptyset$, and $t^0 \equiv 0$.

Step 1: Choose the ordered pair $(a^1 b^1)$ such that

$$(a^1 b^1) = \underset{(ij)\in A^0\times(A^0)^c}{\operatorname{argmin}} c_{ij}.$$

Let $t^1 \equiv \max\{t^0, c_{a^1 b^1}\}$, $A^1 \equiv A^0 \cup \{b^1\}$, and $g^1 \equiv g^0 \cup \{(a^1 b^1)\}$.

Step k: Choose the ordered pair

$$(a^k b^k) = \underset{(ij)\in A^{k-1}\times(A^{k-1})^c}{\operatorname{argmin}} c_{ij}.$$

Let $t^k \equiv \max\{t^{k-1}, c_{a^k b^k}\}$, $A^k \equiv A^{k-1} \cup \{b^k\}$, $g^k \equiv g^{k-1} \cup \{(a^k b^k)\}$, and $\varphi_{k-1}^{DK}(N_0, C) \equiv \min\{t^{k-1}, c_{a^k b^k}\}$.

The algorithm terminates at step n and $\varphi_n^{DK}(N_0, C) \equiv t^n$.

If $C \notin \tilde{\mathcal{C}}$, at some step, several agents might have the same cost and the Prim algorithm may not be able to determine a unique agent. By taking an average of all the possibilities, the DK allocation can be calculated. However, on the domain $\tilde{\mathcal{C}}$, since a unique agent is determined at each step, we do not face such a difficulty.

Since we rename all agents based on the step chosen by the Prim algorithm, for each $i \in N$, the node $b^i$ is agent $i$, and the node $a^i$ is the agent $p(i)$. Thus, for each $i \in N$, $c_{a^i b^i} = c_{p(i)i}$. Therefore, on the domain $\tilde{\mathcal{C}}$, the DK rule can be rewritten as

$$\varphi_k^{DK}(N_0, C) = \min\{\max_{l \leq k}\{c_{p(l)l}\}, c_{p(k+1)k+1}\}, \quad 1 \leq k < n,$$

and $\varphi_n^{DK}(N_0, C) = \max_{l=1,\dots,n} c_{p(l)l}$.

### 2.4 Component-wise efficiency of rules

*Efficiency* requires that the sum of costs assigned to all agents should be equal to the total cost. A stronger requirement of *component-wise efficiency* requires that the sum of costs assigned to each component should be equal to its cost. If a rule does not satisfy *component-wise efficiency,* then there exists some component that would benefit by constructing its own mcst.

**Component-wise efficiency:** For all $N_0 \in \mathcal{N}_0$, all $C \in \mathcal{C}_{N_0}$, all $i \in f(0)$, and all $y \in \mathcal{Y}_N$, a cost allocation $y$ is *component-wise efficient* if

$$\sum_{j \in \{i\} \cup F(i)} y_j = \sum_{j \in \{i\} \cup F(i)} c_{p(j)j}.$$

As shown in Granot and Huberman (1981), any cost allocation in the core satisfies *component-wise efficiency.* Since the Bird and the Dutta-Kar rules are core selections, both of them are *component-wise efficient.*

## 3 Demand operators

In this section, we define two demand operators, the *weak demand operator* (*wdo*) and the *strong demand operator* (*sdo*), and discuss their properties.

### 3.1 The weak demand operator

Granot and Huberman (1984) introduced the *wdo* after considering the opportunity cost of each agent. Suppose that agent $i$ is the immediate predecessor of agent $j$ and each agent pays the cost according to the Bird rule $y = \varphi^B(N_0, C)$.

If agent $i$ leaves, then agent $j$ is connected with the source or another agent and pays more than $y_j$. Therefore, agent $i$ can make a demand for her cooperation with agent $j$ and a transfer of the surplus from agent $j$ to agent $i$ results in a new allocation. When some agent is asking for compensation her followers, we call her the *operating agent.*

For each $N_0 \in \mathcal{N}_0$ and each $C \in \tilde{\mathcal{C}}_{N_0}$, let $t(N_0, C)$ be the *mcst* of the game. For each $k \in N$, a subtree $t_k(N_0, C)$ of $t(N_0, C)$ with root $k$ is defined as $t_k(N_0, C) = \{(ij)|i, j \in \{k\} \cup F(k), (ij) \in t(N_0, C)\}$. Given $t(N_0, C)$, for all $i \in N$, let $\tilde{t}(N_0 \setminus \{i\}, C)$ be an *mcst* for the problem $(N_0 \setminus \{i\}, C)$ such that for all $j \in f(i)$, $t_j(N_0, C) \subseteq \tilde{t}(N_0 \setminus \{i\}, C)$. For each $j \in f(i)$, there exists an arc $(rk)$ such that $(rk)$ is in the unique path from 0 to $j$ in $\tilde{t}(N_0 \setminus \{i\}, C)$, $r \in N_0 \setminus (\{j\} \cup F(j))$, and $k \in \{j\} \cup F(j)$. Let $a^i_j = c_{rk}$.[3]

The *wdo* is a function which associates to each initial allocation a cost allocation obtained after considering how much each agent can demand to her followers.

**Weak Demand Operator** (Granot and Huberman, 1984): For all $N_0 \in \mathcal{N}_0$, all $C \in \tilde{\mathcal{C}}_{N_0}$, all $i \in N$, and all $y \in \mathcal{Y}_N$,

$$wd^i_j(y) = \begin{cases} a^i_j - \sum_{(uv)\in t_j(N_0,C)} c_{uv} + \sum_{l\in F(j)} y_l & \text{if } j \in f(i), \\ y_i - \sum_{k\in f(i)} (wd^i_k(y) - y_k) & \text{if } j = i, \\ y_j & \text{otherwise.} \end{cases}$$

We note that in the definition of the operator the terms after $a^i_j$ vanishes if the Bird allocation is initially given. Also, it is easy to check that $\sum_{j\in N} wd^i_j(y) = \sum_{j\in N} y_j$.

The *wdo* can be interpreted as follows. If agent $i$ makes a demand to her immediate followers, the followers form an alternative *mcst* without $i$ and pay the cost of the alternative tree. Since the alternative tree increases the cost allocation of agent $i$'s immediate followers, the difference can be claimed by agent $i$.

However, the current definition of the *wdo* has a flaw. After agent $i$ leaves, the cost of an alternative tree for $\{j\} \cup F(j)$ will be $a^i_j + \sum_{(uv)\in t_j(N_0,C)} c_{uv}$, but agents in $j \in f(i)$ get a benefit if $\sum_{(uv)\in t_j(N_0,C)} c_{uv}$ increases. Since the cost $a^i_j + \sum_{(uv)\in t_j(N_0,C)} c_{uv}$ should be shared among agents $j$ and her followers, it is awkward to ask only agent $i$ to be responsible for the increase in $\sum_{(uv)\in t_j(N_0,C)} c_{uv}$.

[3]Note that $\tilde{t}(N_0\setminus\{i\}, C)$ may not be unique. Since $c_{rk}$ depends on the choice of $\tilde{t}(N_0\setminus\{i\}, C)$, it may not be uniquely determined either. However, in our application of the demand operators, we will impose a selection rule to make sure that this arc is uniquely determined.

Based on this observation, we suggest the following modification.

**Modified Weak Demand Operator (mwdo)**: For all $N_0 \in \mathcal{N}_0$, all $C \in \tilde{\mathcal{C}}_{N_0}$, all $i \in N$, and $y \in \mathcal{Y}_N$,

$$\overline{wd}^i_j(y) = \begin{cases} (a^i_j + \sum_{(uv)\in t_j(N_0,C)} c_{uv}) - \sum_{l\in F(j)} y_l & \text{if } j \in f(i), \\ y_i - \sum_{k\in f(i)} (\overline{wd}^i_k(y) - y_k) & \text{if } j = i, \\ y_j & \text{otherwise.} \end{cases}$$

As discussed earlier, $(a^i_j + \sum_{(uv)\in t_j(N_0,C)} c_{uv})$ is the cost of the alternative tree for $\{j\} \cup F(j)$, which is constructed after agent $i$ leaves. Thus, in our definition of the *mwdo,* for any $j \in f(i)$, $\{j\} \cup F(j)$ share the cost of an alternative tree. Similarly to the *wdo*, if the Bird allocation is initially given, the terms after $a^i_k$ vanishes in the definition of the operator. Therefore, in this case, the two definitions of the *wdo*'s are the same. Also, as in the case of *wdo,* it is easy to show that $\sum_{j\in N} \overline{wd}^i_j(y) = \sum_{j\in N} y_j$.

## 3.2 The strong demand operator

Suppose an *mcst* is given and agent $i$ is the immediate predecessor of agent $j$. Furthermore, suppose that the initial cost allocation is $y$ and agent $i$ wants to transfer some of her cost to agent $j$. What is the maximum amount that she can transfer? If agent $i$ asks too much, agent $j$ will disconnect the arc $(ij)$ and form her own tree together with other agents. Therefore, agent $i$ can transfer her cost as long as agent $j$ does not have an incentive to form her own tree.

Before we formally define the *sdo*, we introduce one more notation. For $R_1, R_2 \subset N$, a coalition set $T_{R_1,R_2}$ is defined as

$$T_{R_1,R_2} = \{S \subseteq N | \ R_1 \subseteq S, \ R_2 \cap S = \emptyset\}.$$

For convenience, if both $R_1$ and $R_2$ are singletons, say $R_1 = \{i\}$ and $R_2 = \{j\}$, then we write $T_{i,j}$ instead of $T_{\{i\},\{j\}}$.

**Strong Demand Operator** (Granot and Huberman, 1984): To find the maximum amount of the cost transfer, agent $i$ first solves the optimization problem,

$$\max\{\sum_{j\in f(i)} z_j\}$$

subject to $ex(R,z) = c(R) - \sum_{i \in R} z_i \geq 0$ for all $R \in T_{f(i),\{i\}} \cup (T_{S,f(i)\setminus S} : S \subset f(i))$,

$z_k = y_k$ for all $k \notin \{i\} \cup f(i)$, and

$\sum_{i \in N} z_i = \sum_{i \in N} y_i$.

Then, the *sdo* is defined by

$$sd_j^i(y) = \begin{cases} z_j & \text{if } j \in f(i), \\ y_i - \sum_{k \in f(i)} (z_k - y_k) & \text{if } j = i, \\ y_j & \text{otherwise.} \end{cases}$$

We call the first constraints of the optimization problem *participation constraints*. Note that if $y$ is in the core, then the constraints of the *sdo* ensure that all cost allocations in $\{sd^i(y)\}$ are also in the core. The *sdo* measures the maximal amount of transfers that agent $i$ can demand to her followers without giving them an incentive to build their own tree. Also, as in the case of *wdo* and *mwdo*, it is easy to check that $\sum_{j \in N} sd_j^i(y) = \sum_{j \in N} y_j$.

# 4 Main results

In this section, we introduce a procedure to apply the demand operators and show how the DK allocation can be obtained as a consequence.

## 4.1 A procedure for the application of demand operators

We apply the demand operators on the irreducible cost matrix $C^*$ instead of the original cost matrix $C$ for the following two reasons. First, as soon as an *mcst* is constructed, it is difficult to find the meaning of the costs of links which are not a part of the *mcst*. Secondly, on $C$, *mwdo* or *sdo* can assign a negative allocation to the operating agent.

Furthermore, we assume that the operating agent leaves the problem after *mwdo* or *sdo* is applied to her. Since the remaining agents cannot use the operating agent's node, they need to construct an alternative tree. On the irreducible cost matrix, there may be many different ways to construct at each step an *mcst* for the remaining agents. To keep the relationship between the original and the new trees as close as possible, we impose the following selection rule.

**Selection rule $\rho$ for the construction of an mcst:** For all $N_0 \in \mathcal{N}_0$, all $C \in \tilde{\mathcal{C}}_{N_0}$, and all $i \in N$, let agent $\ell \in N$ be such that $\ell \in f(i)$ and for all $j \in f(i) \setminus \{\ell\}$, $\ell < j$. The *alternative mcst*, $\tilde{t}(N_0 \setminus \{i\}, C^*)$, is constructed as follows:

- Agent $\ell$ is connected to $p(i)$ and all other agents $j \in f(i) \setminus \{\ell\}$ are connected to agent $\ell$.

Next we introduce our procedure to apply demand operators sequentially to each agent following the numbering of agents assigned by the Prim algorithm. For all $S \subseteq N$ and all $y \in \mathcal{Y}_N$, let $y_S$ be the projection of $y$ onto $\mathbb{R}^S$.

**Sequential application of a demand operator on the irreducible cost matrix:** For all $N_0 \in \mathcal{N}_0$, all $C \in \tilde{\mathcal{C}}_{N_0}$, and all $y^0 \in \mathcal{Y}_N$,

(i) Transform the cost matrix $C$ into the irreducible cost matrix $C^*$.

(ii) Let $N^1 = N$ and $y^1 = y^0$ be the initial cost allocation.

(iii) At Step $k = 1, \ldots, n$, we apply a demand operator[4] to agent $k$ and obtain a new cost allocation $z^k \in \mathcal{Y}_{N^k}$.[5]

(iv) Agent $k$ leaves and the remaining agents construct an alternative tree following the selection rule $\rho$.

(v) The procedure repeats with $N^{k+1} = N^k \setminus \{k\}$ and the initial cost allocation $y^{k+1} = z^k_{N^k \setminus \{k\}}$.

## 4.2 The modified weak demand operator and the Dutta-Kar rule

Now we are ready to present our first main result on the relation between the *mwdo* and the Dutta-Kar rule. In particular, we show that if the initial allocation is *component-wise efficient,* then the Dutta-Kar allocation is obtained after applying the *mwdo* sequentially to all agents.

First, we determine how much agent 1 should pay when the *mwdo* is applied to her. If the initial cost allocation is component-wise efficient, then she pays the minimum of the two costs, her connection cost and agent 2's connection cost.

[4] If necessary, we use the selection rule $\rho$.

[5] At Step $n$, the procedure ends at (iii).

**Lemma 1.** *For all $N_0 \in \mathcal{N}_0$, all $C \in \tilde{\mathcal{C}}_{N_0}$, and all $y \in \mathcal{Y}_N$, if $y$ is component-wise efficient, then $\overline{wd}_1^1(y) = \min\{c^*_{01}, c_{p(2)2}\}$.*

*Proof.* Let $N_0 \in \mathcal{N}_0$, $C \in \tilde{\mathcal{C}}_{N_0}$, and $y$ be the initial cost allocation which is component-wise efficient. Let $\overline{wd}^1(y)$ be the new cost allocation obtained after the *mwdo* is applied to agent 1.

The proof is divided into two cases.

**Case I.** $f(1) = \emptyset$: In this case, agent 1 is the only agent in the problem or all other agents belong to different components. Since $y$ is component-wise efficient and agent 1 does not have any follower to transfer her cost, she pays $c^*_{01}$. Since agent 2 belongs to a different component, $c^*_{01} < c_{p(2)2}$. Altogether, $\overline{wd}_1^1(y) = c^*_{01} = \min\{c^*_{01}, c_{p(2)2}\}$.

**Case II.** $f(1) \neq \emptyset$: From the definition of the *mwdo*,

$$\begin{aligned}
\overline{wd}_1^1(y) &= y_1 - \sum_{s \in f(1)} (\overline{wd}_s^1(y) - y_s) \\
&= y_1 - \sum_{s \in f(1)} ((a_s^1 + \sum_{(ij) \in t_s(N_0,C)} c^*_{ij} - \sum_{r \in F(s)} y_r) - y_s) \\
&= (y_1 + \sum_{s \in f(1)} \sum_{r \in F(s)} y_r + \sum_{s \in f(1)} y_s) - \sum_{s \in f(1)} a_s^1 - \sum_{s \in f(1)} \sum_{(ij) \in t_s(N_0,C)} c^*_{ij} \\
&= \sum_{s \in \{1\} \cup F(1)} y_s - \sum_{s \in f(1)} a_s^1 - \sum_{s \in f(1)} \sum_{(ij) \in t_s(N_0,C)} c^*_{ij}.
\end{aligned}$$

Let agent $\ell$ be such that $\ell \in f(1)$ and for all $m \in f(1) \setminus \{\ell\}$, $\ell < m$. From the definition of the selection rule $\rho$, $a_\ell^1 = \max\{c^*_{01}, c^*_{1\ell}\}$ and for all $s \in f(1) \setminus \{\ell\}$, $a_s^1 = c_{p(s)s}$. Together with the definition of $t_s(N_0, C)$, we have

$$\begin{aligned}
\overline{wd}_1^1(y) = & \sum_{s \in \{1\} \cup F(1)} y_s - \sum_{s \in f(1)} a_s^1 - \sum_{s \in f(1)} \sum_{(ij) \in t_s(N_0,C)} c^*_{ij} \\
= & \sum_{s \in \{1\} \cup F(1)} y_s - \max\{c^*_{01}, c^*_{1\ell}\} - \sum_{s \in f(1) \setminus \{\ell\}} c_{p(s)s} - \sum_{s \in f(1)} \sum_{(ij) \in t_s(N_0,C)} c^*_{ij} \\
= & \sum_{s \in \{1\} \cup F(1)} y_s - \max\{c^*_{01}, c^*_{1\ell}\} - \sum_{s \in f(1) \setminus \{\ell\}} c_{p(s)s} - \sum_{s \in f(1)} \sum_{r \in F(s)} c_{p(r)r}.
\end{aligned}$$

Since $y$ is *component-wise efficient,* $\sum_{s\in\{1\}\cup F(1)} y_s = \sum_{s\in\{1\}\cup F(1)} c_{p(s)s}$. Therefore,

$$\begin{aligned}
\overline{wd}_1^1(y) &= \sum_{s\in\{1\}\cup F(1)} y_s - \max\{c^*_{01}, c^*_{1\ell}\} - \sum_{s\in f(1)\setminus\{\ell\}} c_{p(s)s} - \sum_{s\in f(1)} \sum_{r\in F(s)} c_{p(r)r} \\
&= \sum_{s\in\{1\}\cup F(1)} c_{p(s)s} - \max\{c^*_{01}, c^*_{1\ell}\} - \sum_{s\in f(1)\setminus\{\ell\}} c_{p(s)s} - \sum_{s\in f(1)} \sum_{r\in F(s)} c_{p(r)r} \\
&= c_{p(1)1} + c_{p(\ell)\ell} - \max\{c^*_{01}, c^*_{1\ell}\} \\
&= c^*_{01} + c^*_{1\ell} - \max\{c^*_{01}, c^*_{1\ell}\} \\
&= \min\{c^*_{01}, c^*_{1\ell}\}.
\end{aligned}$$

Since either $c^*_{01} < c_{p(2)2} \le c^*_{1\ell}$ or $c^*_{01} \ge c_{p(2)2} = c^*_{1\ell}$,[6] $\min\{c^*_{01}, c^*_{1\ell}\} = \min\{c^*_{01}, c_{p(2)2}\}$. Therefore, $\overline{wd}_1^1(y) = \min\{c^*_{01}, c_{p(2)2}\}$.

Altogether, we conclude that $\overline{wd}_1^1(y) = \min\{c^*_{01}, c_{p(2)2}\}$. □

Suppose that we begin with a component-wise efficient allocation and apply the *mwdo* to the first agent. After agent 1 leaves, $\overline{wd}^1_{N\setminus\{1\}}(y)$ is the cost allocation of the remaining agents. Next we show that this allocation is component-wise efficient for the alternative mcst of the remaining agents.

**Lemma 2.** *For all $N_0 \in \mathcal{N}_0$, all $C \in \tilde{\mathcal{C}}_{N_0}$, and all $y \in \mathcal{Y}_N$, if $y$ is component-wise efficient, then $\overline{wd}^1_{N\setminus\{1\}}(y)$ is component-wise efficient for the alternative mcst $\tilde{t}(N_0 \setminus \{1\}, C^*)$.*

*Proof.* Let $N_0 \in \mathcal{N}_0$, $C \in \tilde{\mathcal{C}}_{N_0}$, and $y$ be the initial cost allocation which is component-wise efficient. Since the *mwdo* does not affect the cost allocation of agents belonging to other components, all we have to check is how the allocation is affected in the component the operating agent belongs to.

Once again, the proof is divided into two cases.

**Case I.** $f(1) = \emptyset$: Since agent $1$ is the only agent in the component, this component will be empty after the *mwdo* is applied to agent $1$. Since $y$ is component-wise efficient, $\overline{wd}^1_{N\setminus\{1\}}(y)$ is component-wise efficient for the alternative *mcst*.

[6] In either case, the equality in the second relation holds when $\ell$= 2.

**Case II.** $f(1) \neq \emptyset$: From the definition of the *mwdo*, $\sum_{s\in\{1\}\cup F(1)} \overline{wd}^1_s(y) = \sum_{s\in\{1\}\cup F(1)} y_s$. Since $y$ satisfies component-wise efficiency, $\sum_{s\in\{1\}\cup F(1)} y_s = \sum_{s\in\{1\}\cup F(1)} c_{p(s)s}$. By Lemma 1, $\overline{wd}^1_1(y) = \min\{c^*_{01}, c_{p(2)2}\}$.

Let agent $\ell$ be such that $\ell \in f(1)$ and for all $m \in f(1)\setminus\{\ell\}$, $\ell < m$. From the definition of the selection rule $\rho$, $a^1_\ell = \max\{c^*_{01}, c^*_{1\ell}\}$ and for all $s \in f(1)\setminus\{\ell\}$, $a^1_s = c_{p(s)s}$. Since $a^1_\ell$ is the cost of the arc $(0\ell)$ after agent 1 leaves, the cost of the alternative tree for this component is $\sum_{s\in F(1)\setminus\{\ell\}} c_{p(s)s} + a^1_\ell = \sum_{s\in F(1)\setminus\{\ell\}} c_{p(s)s} + \max\{c^*_{01}, c^*_{1\ell}\}$.

After agent 1 leaves,

$$\begin{aligned}
\sum_{s\in F(1)} \overline{wd}^1_s(y) &= \sum_{s\in\{1\}\cup F(1)} \overline{wd}^1_s(y) - \overline{wd}^1_1(y)\\
&= \sum_{s\in\{1\}\cup F(1)} y_s - \overline{wd}^1_1(y)\\
&= \sum_{s\in\{1\}\cup F(1)} c_{p(s)s} - \overline{wd}^1_1(y)\\
&= \sum_{s\in\{1\}\cup F(1)} c_{p(s)s} - \min\{c^*_{01}, c_{p(2)2}\}\\
&= \sum_{s\in F(1)\setminus\{\ell\}} c_{p(s)s} + c^*_{01} + c^*_{1\ell} - \min\{c^*_{01}, c_{p(2)2}\}\\
&= \sum_{s\in F(1)\setminus\{\ell\}} c_{p(s)s} + c^*_{01} + c^*_{1\ell} - \min\{c^*_{01}, c^*_{1\ell}\}\\
&= \sum_{s\in F(1)\setminus\{\ell\}} c_{p(s)s} + \max\{c^*_{01}, c^*_{1\ell}\}.
\end{aligned}$$

Therefore, $\overline{wd}^1_{N\setminus\{1\}}(y)$ is component-wise efficient for the alternative *mcst*.

Altogether, we conclude that $\overline{wd}^1_{N\setminus\{1\}}(y)$ is component-wise efficient for $\tilde{t}(N_0\setminus\{1\}, C^*)$. $\square$

Now we are ready to present our first main result.

**Theorem 1.** *For all $N_0 \in \mathcal{N}_0$, all $C \in \tilde{\mathcal{C}}_{N_0}$, and all $y \in \mathcal{Y}_N$, if $y$ is component-wise efficient, then the allocation obtained by applying the mwdo from $y$ coincides with the DK allocation.*

*Proof.* Let $N_0 \in \mathcal{N}_0$, $C \in \tilde{\mathcal{C}}_{N_0}$, and $y^0$ be the initial cost allocation which is component-wise efficient. Let $N^1 = N$ and $y^1 = y^0$. At Step 1, the *mwdo* is applied to agent 1. Let $z^1 = \overline{wd}^1(y^1)$. By Lemma 1, $z^1_1 = \min\{c^*_{01}, c_{p(2)2}\} = \min\{c_{01}, c_{p(2)2}\}$. After that, $N^2 = N^1 \setminus \{1\}$ constructs an alternative tree according to the selection rule $\rho$. Let $y^2 = z^1_{N^2}$. By Lemma 2, $y^2$ is component-wise efficient for the alternative *mcst*. The selection rule ensures that $2 \in f(0)$ with the connection cost $c^*_{02} = \max\{c_{01}, c_{p(2)2}\}$.

At Step 2, the *mwdo* is applied to agent 2 and similarly, $z^2 = \overline{wd}^2(y^2)$. By Lemma 1, $z^2_2 = \min\{c^*_{02}, c_{p(3)3}\}$, and by Lemma 2, $y^3 = z^2_{N^3}$ is component-wise efficient for the alternative *mcst* with $N^3 = N^2 \setminus \{2\}$. The selection rule ensures that $3 \in f(0)$ with the connection cost $c^*_{03} = \max\{c^*_{02}, c_{p(3)3}\} = \max\{c_{01}, c_{p(2)2}, c_{p(3)3}\}$. $\ldots$ Let $z^k = \overline{wd}^k(y^k)$, $y^{k+1} = z^k_{N^{k+1}}$, and $N^{k+1} = N^k \setminus \{k\}$.

At Step $k$ $(k < n)$, the *mwdo* is applied to agent $k$, and $z^k = \overline{wd}^k(y^k)$. Once again, note that $z^k_k = \min\{c^*_{0k}, c_{p(k+1)(k+1)}\}$, $c^*_{0k} = \max\{c_{01}, c_{p(2)2}, \ldots, c_{p(k)k}\} = \max_{j\leq k}\{c_{p(j)j}\}$, and $y^{k+1} = z^k_{N^{k+1}}$ is component-wise efficient for the alternative *mcst*.

At the last step, agent $n$ pays $c^*_{0n} = \max_{j=1,\ldots,n}\{c_{p(j)j}\}$.

Therefore, at each step, the operating agent $k < n$ pays $\overline{wd}^k_k(y^k) = \min\{\max_{l\leq k}\{c_{p(l)l}\}, c_{p(k+1)k+1}\}$ which coincides with $\varphi^{DK}_k(N_0, C)$ and agent $n$ pays $\max_{j=1,\ldots,n}\{c_{p(j)j}\}$ which coincides with $\varphi^{DK}_n(N_0, C)$. □

## 4.3 The strong demand operator and the Dutta-Kar rule

We discuss the relation between the *sdo* and the Dutta-Kar allocation. In particular, we show that if the initial allocation is in the irreducible core, then the Dutta-Kar allocation is obtained after applying the *sdo* sequentially to all agents.

First, we determine how much agent 1 should pay when the sdo is applied to her. If the initial cost allocation is in the irreducible core, then she pays the minimum of the two costs, her connection cost and agent 2's connection cost, the same as the *mwdo*.

**Lemma 3.** *For all $N_0 \in \mathcal{N}_0$, all $C \in \tilde{\mathcal{C}}_{N_0}$, and all $y \in \mathcal{Y}_N$, if $y \in Core(N, c^*)$, then $sd^1_1(y) = \min\{c^*_{01}, c_{p(2)2}\}$.*

*Proof.* Let $N_0 \in \mathcal{N}_0$, $C \in \tilde{\mathcal{C}}_{N_0}$, and $y \in Core(N, c^*)$ be the initial cost allocation. Let $sd^1(y)$ be the new cost allocation obtained after the *sdo* is applied to agent 1. As Granot and Huberman (1984) mentioned, if an initial allocation is in the core, then the constraints of the *sdo* ensure that the outcome is in the core. Therefore, it is obvious that $sd^1(y) \in Core(N, c^*)$.

We divide the proof into two cases:

**Case I.** $f(1) = \emptyset$: In this case, agent 1 is the only agent in the problem or all other agents belong to different components. Since $y$ is in the irreducible core, it is component-wise efficient. Moreover, agent 1 does not have any followers to transfer her cost. Therefore, agent 1 should pay $c^*_{01}$. Since agent 2 belongs to different components, $c^*_{01} < c_{p(2)2} = c_{02}$. Altogether, $sd^1_1(y) = c^*_{01} = \min\{c^*_{01}, c_{p(2)2}\}$.

**Case II.** $f(1) \neq \emptyset$: Let agent $\ell$ be such that $\ell \in f(1)$ and for all $m \in f(1) \setminus \{\ell\}$, $\ell < m$. Note that $c^*(N \setminus \{1\}) = c^*(N) - \min\{c^*_{01}, c^*_{1\ell}\}$. Since either $c^*_{01} < c_{p(2)2} \leq c^*_{1\ell}$ or $c^*_{01} \geq c_{p(2)2} = c^*_{1\ell}$,[7] $\min\{c^*_{01}, c^*_{1\ell}\} = \min\{c^*_{01}, c_{p(2)2}\}$. Since $y \in Core(N, c^*)$, $c^*(N) = \sum_{i \in N} y_i$. From the definition of the *sdo*, $\sum_{i \in N} sd^1_i(y) = \sum_{i \in N} y_i$.

First, suppose that $sd^1_1(y) < \min\{c^*_{01}, c_{p(2)2}\}$. Then,

$$
\begin{aligned}
ex(N \setminus \{1\}, sd^1(y)) &= c^*(N \setminus \{1\}) - \sum_{j \in N \setminus \{1\}} sd^1_j(y) \\
&= (c^*(N) - \min\{c^*_{01}, c^*_{1l}\}) - (\sum_{j \in N} sd^1_j(y) - sd^1_1(y)) \\
&= (c^*(N) - \sum_{j \in N} sd^1_j(y)) - (\min\{c^*_{01}, c^*_{1l}\} - sd^1_1(y)) \\
&= -\min\{c^*_{01}, c^*_{1l}\} + sd^1_1(y) \\
&= -\min\{c^*_{01}, c_{p(2)2}\} + sd^1_1(y) \\
&< 0.
\end{aligned}
$$

Since $sd^1(y)$ is in the irreducible core, each coalition should have a non-negative excess under $sd^1(y)$, a contradiction.

Secondly, suppose that $sd^1_1(y) > \min\{c^*_{01}, c_{p(2)2}\}$. Let $\epsilon = sd^1_1(y) - \min\{c^*_{01}, c_{p(2)2}\} > 0$. Since $sd^1(y)$ is in the irreducible core, for all $S \subseteq N$, $ex(S, sd^1(y)) = c^*(S) - \sum_{j \in S} sd^1_j(y) \geq 0$. Let $\hat{y}$ be a new cost allocation obtained from $sd^1(y)$ after agent 1 transfers $\epsilon$ to agent $\ell$. We consider four cases: for all coalitions $R \subset N$,

(1) $R \cap \{1, \ell\} = \emptyset$: $ex(R, \hat{y}) \geq 0$ since $\forall i \in N \setminus \{1, \ell\},\ \hat{y}_i = sd^1_i(y)$.

[7] In either case, the equality in the second relation holds when $\ell = 2$.

(2) $R \cap \{1, \ell\} = \{1\}$: $ex(R, \hat{y}) > 0$ since $\forall i \in N \setminus \{1, \ell\}, \hat{y}_i = sd_i^1(y)$ and $\hat{y}_1 < sd_1^1(y)$.

(3) $R \cap \{1, \ell\} = \{1, \ell\}$: $ex(R, \hat{y}) \geq 0$ since $\sum_{i \in R} \hat{y}_i = \sum_{i \in R} sd_i^1(y)$.

(4) $R \cap \{1, \ell\} = \{\ell\}$: Let $R^{+1} \equiv R \cup \{1\}$. Since $sd^1(y)$ is in the irreducible core, $ex(R^{+1}, sd^1(y)) \geq 0$. Also, note that $c^*(R) = c^*(R^{+1}) - \min\{c_{01}^*, c_{1\ell}^*\}$. Then,

$$
\begin{aligned}
ex(R, \hat{y}) &= c^*(R) - \sum_{i \in R} \hat{y}_i \\
&= c^*(R^{+1}) - \min\{c_{01}^*, c_{1\ell}^*\} - \{\sum_{i \in R^{+1}} \hat{y}_i - \hat{y}_1\} \\
&= c^*(R^{+1}) - \min\{c_{01}^*, c_{1\ell}^*\} - \{\sum_{i \in R^{+1}} sd_i^1(y) - \hat{y}_1\} \\
&= c^*(R^{+1}) - \sum_{i \in R^{+1}} sd_i^1(y) + \hat{y}_1 - \min\{c_{01}^*, c_{1\ell}^*\} \\
&= c^*(R^{+1}) - \sum_{i \in R^{+1}} sd_i^1(y) + sd_1^1(y) - \epsilon - \min\{c_{01}^*, c_{1\ell}^*\} \\
&= c^*(R^{+1}) - \sum_{i \in R^{+1}} sd_i^1(y) + \min\{c_{01}^*, c_{p(2)2}\} - \min\{c_{01}^*, c_{1\ell}^*\} \\
&= c^*(R^{+1}) - \sum_{i \in R^{+1}} sd_i^1(y) \\
&\geq 0.
\end{aligned}
$$

Since agent 1 can transfer $\epsilon$ to agent $\ell$ without violating the participation constraints, $sd^1(y)$ cannot be a solution of the optimization problem for the *sdo,* a contradiction.

Altogether, we conclude that $sd_1^1(y) = \min\{c_{01}^*, c_{p(2)2}\}$. $\square$

Suppose that we begin with an initial allocation $y$ in the irreducible core and apply the *sdo* to the first agent. After agent 1 leaves, $sd_{N \setminus \{1\}}^1(y)$ is the cost allocation assigned to the remaining agents. Next we show that this allocation is in the irreducible core of the problem for the remaining agents.

**Lemma 4.** *For all* $N_0 \in \mathcal{N}_0$, *all* $C \in \tilde{\mathcal{C}}_{N_0}$, *and all* $y \in \mathcal{Y}_N$, *if* $y \in Core(N, c^*)$, *then* $sd_{N \setminus \{1\}}^1(y) \in Core(N \setminus \{1\}, c^*)$.

*Proof.* Let $N_0 \in \mathcal{N}_0$, $C \in \tilde{\mathcal{C}}_{N_0}$, $y \in Core(N, c^*)$, and $1 \in f(0)$. First, we show that the excess conditions are satisfied. Since $y \in Core(N, c^*)$, $sd^1(y) \in Core(N, c^*)$. Therefore, in the *mcstp* $(N_0 \setminus \{1\}, C^*)$, for all $R \subseteq N \setminus \{1\}$, $ex(R, sd^1_{N\setminus\{1\}}(y)) \geq 0$.

Next we show that the *efficiency* condition is satisfied. From the definition of the *sdo,* $\sum_{j\in N} y_j = \sum_{j\in N} sd^1_j(y)$. Also, since $y$ is in the irreducible core, $\sum_{i\in N} y_i = \sum_{(ij)\in t(N_0,C^*)} c^*_{ij}$. By the selection rule $\rho$,

$$\sum_{(ij)\in t(N_0,C^*)} c^*_{ij} = \sum_{(ij)\in \tilde{t}(N_0\setminus\{1\},C^*)} c^*_{ij} + \min\{c^*_{01}, c_{p(2)2}\}$$

and by Lemma 3,

$$\sum_{i\in N} sd^1_i(y) = \sum_{i\in N\setminus\{1\}} sd^1_i(y) + sd^1_1(y) = \sum_{i\in N\setminus\{1\}} sd^1_i(y) + \min\{c^*_{01}, c_{p(2)2}\}.$$

Therefore, $\sum_{(ij)\in \tilde{t}(N_0\setminus\{1\},C^*)} c^*_{ij} = \sum_{i\in N\setminus\{1\}} sd^1_i(y)$, as desired.

Altogether, we conclude that $sd^1_{N\setminus\{1\}}(y) \in Core(N \setminus \{1\}, c^*)$. □

Now we are ready to present our second main result.

**Theorem 2.** *For all $N_0 \in \mathcal{N}_0$, all $C \in \tilde{\mathcal{C}}_{N_0}$, and all $y \in \mathcal{Y}_N$, if $y \in Core(N, c^*)$, then the allocation obtained by applying the sdo from $y$ coincides with the DK allocation.*

*Proof.* Let $N_0 \in \mathcal{N}_0$, $C \in \tilde{\mathcal{C}}_{N_0}$, and $y^0 \in Core(N, c^*)$ be the initial cost allocation. Let $N^1 = N$ and $y^1 = y^0$. At Step 1, the *sdo* is applied to agent 1. Let $z^1 = sd^1(y^1)$. By Lemma 3, $z^1_1 = \min\{c^*_{01}, c_{p(2)2}\} = \min\{c_{01}, c_{p(2)2}\}$. After that, $N^2 = N^1 \setminus \{1\}$ constructs an alternative tree according to the selection rule $\rho$. By Lemma 4, $y^2 = z^1_{N^2} \in Core(N^2, c^*)$. The selection rule ensures that $2 \in f(0)$ with the connection cost $c^*_{02} = \max\{c_{01}, c_{p(2)2}\}$.

At Step 2, the *sdo* is applied to agent 2 and similarly, $z^2 = sd^2(y^2)$. By Lemma 3, $z^2_2 = \min\{c^*_{02}, c_{p(3)3}\}$, and Lemma 4, $y^3 = z^2_{N^3} \in Core(N^3, c^*)$. The selection rule ensures that $3 \in f(0)$ with the connection cost $c^*_{03} = \max\{c^*_{02}, c_{p(3)3}\} = \max\{c_{01}, c_{p(2)2}, c_{p(3)3}\}$. ... Let $z^k = sd^k(y^k)$, $y^{k+1} = z^k_{N^{k+1}}$, and $N^{k+1} = N^k \setminus \{k\}$.

At Step $k$ $(k < n)$, the *sdo* is applied to agent $k$, $z^k = sd^k(y^k)$. Once again, note that $z^k_k = \min\{c^*_{0k}, c_{p(k+1)(k+1)}\}$, $c^*_{0k} = \max\{c_{01}, c_{p(2)2}, \ldots, c_{p(k)k}\} = \max_{j\leq k}\{c_{p(j)j}\}$, and $y^{k+1} = z^k_{N^{k+1}} \in Core(N^{k+1}, c^*)$.

At the last step, agent $n$ pays $c^*_{0n} = \max_{j=1,\dots,n}\{c_{p(j)j}\}$.

Therefore, at each step, the operating agent $k < n$ pays $sd^k_k(y^k) = \min\{\max_{l\leq k}\{c_{p(l)l}\}, c_{p(k+1)k+1}\}$ which coincides with $\varphi^{DK}_k(N_0, C)$ and agent $n$ pays $\max_{j=1,\dots,n}\{c_{p(j)j}\}$ which coincides with $\varphi^{DK}_n(N_0, C)$. $\square$

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