Discussion Papers in Economics

Hierarchical Outcomes and Collusion Neutrality

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Discussion Paper No. 87
December 2012

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December 27, 2012

Abstract

We consider collusive agreements in (superadditive TU-)games that bind collusion members to act as a single player in such a way that the collusion performs its role in the original game if all members are together; otherwise, the collusion plays no role, acting like the empty set. Collusive agreements are feasible only for a set of players who are connected on a tree. *Collusion neutrality* requires that no feasible collusive agreement influences the total payoff for the collusion members. On the domain of all games, tree-restricted or unrestricted, there is a solution satisfying *collusion neutrality*, *efficiency* and *null-player property* if and only if the tree is a line. Either replacing *null-player property* with *very-null-player property* or restricting the domain to the subdomain of tree-restricted games, we show that affine combinations of hierarchical solutions (Demange 2004, van den Brink 2012) are the only solutions satisfying the three axioms together with *linearity*. All these results hold replacing *collusion neutrality* with the combination of *pairwise neutrality* and *non-bossiness*. Adding a mild equal treatment axiom, we obtain characterizations of the average tree solution (the average of hierarchical solutions, i.e., the affine combination with equal weights).

*Acknowledgements will be added in the final version.

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JEL classification: C71

Keywords: Hierarchical outcomes, Collusion neutrality, TU-game, Efficiency, Null-player property, Tree, Network

1 Introduction

Collusive agreements among some players in a cooperative game may influence the structure of the game, and as a result, the total payoff of the collusion members may change. This possibility has been raised by Harsanyi (1977) through the joint-bargaining paradox in a bargaining framework; collusive agreements can weaken the bargaining position of the collusion members. The best known solution, the Shapley value, exhibits the same paradoxical feature (Haller 1994). Three different types of collusive agreements have been studied by numerous authors; see Haller (1994), Malawski (2002), van den Brink (2009, 2012), among others. Here we consider collusive agreements that bind collusion members to act as a single player in such a way that the collusion performs its role in the original game if all members are together; otherwise, the collusion plays no role, acting like the empty set (Malawski 2002, van den Brink 2009, 2012). Our key axiom, *collusion neutrality*, requires that no coalition can influence the total payoff of its members through the collusive agreements.

The most studied solution that is neutral to collusive agreements is the Banzhaf value. Unfortunately, it is not efficient. The characterization of the Banzhaf value in Malawski (2002) shows that no efficient and neutral solution can be found as long as we also require the other standard axioms in the characterization including *null-player property*. van den Brink (2012) establishes a stronger non-existence result that there is no solution satisfying *collusion neutrality* as well as *efficiency* and *null-player property*. However, for "tree-restricted games", namely, games with a restricted coalition formation on a tree network, van den Brink (2012) shows that there do exist solutions satisfying the three axioms. Nevertheless, there remain two crucial open questions in this line of research.

In van den Brink (2012), both collusive agreements and coalition formation are subject to the same network constraint. Since collusive agreements require strong commitments of members in addition to forming the coalition itself, one may well argue that the constraint for collusive agreements could be stricter than the constraint for coalition formations. This leads to the first question. Will the existence result in van den Brink (2012) go through with different constraints for the collusive agreements and for coalition formation? Of particular interest in our investigation is the environment where only collusive agreements are subject to a network constraint and there is no constraint in coalition formation. The second question is about the uniqueness. Are the solutions in van den Brink (2012) unique ones satisfying the three axioms? Our goal is to answer these two questions.

As in Demange (1994, 2004), Herings et al. (2008), and van den Brink (2012), we consider a tree network constraint. We assume that collusive agreements are feasible for collusions that are connected on the tree. Unlike these three papers, however, there is no restriction on coalition formations. That is, we consider all superadditive games, tree-restricted or unrestricted. The model considered by van den Brink (2012), in our investigation, corresponds to a special case, the subset of tree-restricted games. Our first result shows that the existence of solutions satisfying *collusion neutrality*, *efficiency* and *null-player property* is guaranteed if and only if the tree is a line. Therefore, on the domain of all superadditive games, with or without the tree-restriction, the existence result of van den Brink (2012) no longer holds except when the tree is a line.

Interpreting a rooted tree as a hierarchy, the hierarchical solution (Demange 2004) allows each "subtree" coalition, consisting of a player and all her successors, to get its worth as its total payoff. All hierarchical solutions satisfy *collusion neutrality* and *efficiency*. They violate *null-player property*, but the violation never occurs on tree-restricted games (van den Brink 2012). Our second result shows that affine combinations (linear combinations the sum of which coefficients equals 1) of these hierarchical solutions are the only solutions satisfying the three axioms and linearity on the domain of tree-restricted games. Linearity is the combination of homogeneity (of degree 1) and additivity and is used in other characterization results by Shapley (1953), Lehrer (1988), Haller (1994), van den Brink (2009), Mishra and Talman (2010), etc. The second result also holds on the domain of all superadditive games when *null-player property* is replaced with a weaker axiom called *very-null-player property*.

All our main results continue to hold when *collusion neutrality* is replaced with a weaker axiom, *pairwise neutrality* (pertaining to collusions of two players) and *non-bossiness* (pairwise collusive agreement should not influence the payoffs of other players). As corollaries, we obtain characterizations of the average tree solution (the average of hierarchical solutions, i.e., the affine combination with equal weights) by van den Brink (2009) and Mishra and Talman (2010).

The hierarchical solutions we characterize here provide stable outcomes in the core for all superadditive tree-restricted games, as shown by Demange (2004). In fact, hierarchical outcomes are extreme points of the core. Earlier in this line of research, Kaneko and Wooders (1982), Le Breton et al. (1992), and Demange (1994) show that the core of a tree-restricted game is not empty for all superadditive games. The application of the Shapley value to network games (including tree-restricted games) is known as Myerson value (Myerson 1977). It selects a core allocation not always but only when a network game is convex.

2 Model and Preliminaries

A cooperative game with transferable utility, briefly a game is defined by the set of players $N \equiv \{1, 2, \dots, n\}$ and the characteristic function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, which associates with each non-empty coalition $S \subseteq N$ the total payoff $v(S)$ that the coalition can produce. The set of players N is fixed and a game is denoted by the characteristic function v. For example, for all non-empty coalitions $T \subseteq N$, *T-unanimity game* is denoted by the characteristic function u^T defined as follows: for all $S \subseteq N$, if $T \subseteq S$, $u^T(S) = 1$; otherwise, $u^T(S) = 0$.

Game v is *superadditive* if for all disjoint $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T)$. Let $\mathcal{V}$ be the collection of all superadditive games. A payoff vector $x \equiv (x_i)_{i \in N} \in \mathbb{R}^N$ is an n-dimensional vector, where x_i is the payoff of player $i \in N$. A *solution* $f : \mathcal{V} \to \mathbb{R}^N$ associates with each game v a payoff vector $f(v) \in \mathbb{R}^N$. The following three axioms for solutions are standard in the cooperative game literature.

For superadditive games, grand coalition N gives the greatest opportunities for all players. Thus efficiency here can be stated as follows:

Efficiency. For all $v \in \mathcal{V}$, $\sum_{i \in N} f_i(v) = v(N)$.

A *null-player* $i \in N$ of game v is a player who has no contribution to others, that is, for all $S \subseteq N \backslash \{i\}$, $v(S \cup \{i\}) = v(S)$. Let $null(v)$ be the set of null players in v. The next axiom requires that null players should not be rewarded.

Null-Player Property. For all $v \in \mathcal{V}$, if $i \in N$ is a null player in v (i.e. $i \in null(v)$), then $f_i(v) = 0$.

Next is linearity considered by numerous authors in the cooperative game literature after Shapley (1953), in particular, by Lehrer (1988), Haller (1994), van den Brink (2009), Mishra and Talman (2010), closely related with our research.

Linearity. For all $v, w \in \mathcal{V}$ and all $c \geq 0$, $f_i(v) + f_i(w) = f_i(v + w)$ and $f_i(cv) = cf_i(v)$.

Players can make a collusive agreement that bind the collusion members to act as a single player in such a way that the collusion performs its role in the original game if all members are together; otherwise, the collusion plays no role, acting like the empty set. We look for solutions that are "neutral" to such collusive agreements so that no collusion can increase or decrease its total payoff.

When there is no restriction on the collusion formation, that is, any subset of players can reach a collusive agreement, it is shown that there is no solution satisfying the neutrality property together with *efficiency* and *null-player property* (van den Brink 2012). Here we assume that the collusive agreements can be made over a network and only connected coalitions are feasible collusions.

A network is described by a set of edges $L \subseteq N \times N$. For all $i, j \in N$ with $i \neq j$, a *path* from i to j is a sequence of nodes $(i_1, \ldots, i_k)$ such that $i_1 = i$, $i_k = j$, and $(i_1, i_2) \in L, \ldots, (i_{k-1}, i_k) \in L$. Network L is a *tree*, if for all $i, j \in N$ with $i \neq j$, there is a unique path from i to j.

In what follows, *assume that* (N, L) *is a tree.* Let $\mathcal{C}(L)$ be the set of connected coalitions on the tree L. Given a game v, *collusive agreements* among members of a connected coalition $T \in \mathcal{C}(L)$ enforce each member to act as she does in the original game if she is with all other members, and to play no role if any other member is missing. Hence a *collusion* of $T \in \mathcal{C}(L)$ turns the game v into v_T such that for all $S \subseteq N$, if $T \nsubseteq S$, $v_T(S) = v(S \backslash T)$; if $T \subseteq S$, $v_T(S) = v(S)$. Our main axiom requires that for any *feasible collusion on the tree*, the total payoff of the collusion members should not be influenced by their collusive agreement.

Collusion Neutrality. For all connected coalition $T \subseteq N$ on L, $\sum_{i \in T} f_i(v_T) = \sum_{i \in T} f_i(v)$.

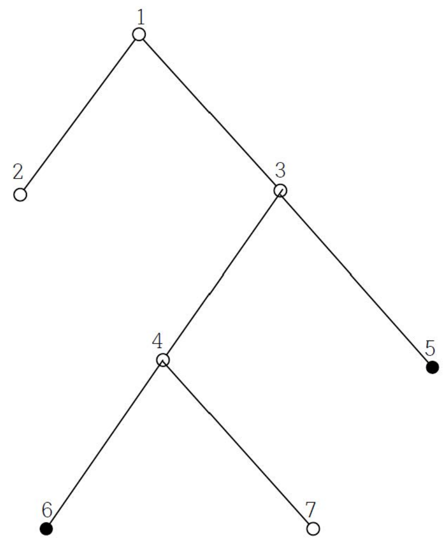

Figure 1: When there are two non-null players 5 and 6 (i.e. $null(v) \equiv \{1, 2, 3, 4, 7\}$) on the tree, 3 and 4 are null players but they are not very null players. All other null players $(1, 2, 7)$ are very null players.

Pairwise neutrality (Malawski 2002) is defined as above focusing on two-player coalitions, namely edges.

Despite the tree-restriction on collusive agreements, we will show that except when L is a line, there is no solution satisfying *collusion neutrality* (or *pairwise neutrality*) together with *null-player-property* and *efficiency*. A slight weakening of *null-player-property*, however, leads to a positive result. We drop the requirement of giving the zero payoff to some null players who can play a critical role in "connecting two non-null players" so that they can make a collusive agreement. All other null players should receive the zero payoff. Given a tree (N, L), for all $T \subseteq N$, let T^{cn} be the smallest connected set including T (it is unique in the case of a tree). A player $i \in N$ is a *very null player* if she is a null player and cannot connect two non-null players, that is, $i \notin [N \backslash null(v)]^{cn}$. For example, on the tree in Figure 1, when all players except for 5 and 6 are null players, $[N \backslash null(v)]^{cn} = \{3, 4, 5, 6\}$ and 3 and 4 are null players but not very null players.

Very-Null-Player Property. For all $v \in \mathcal{V}$, if $i \in N$ is a very null player in v (i.e. $i \notin [N \backslash null(v)]^{cn}$), then $f_i(v) = 0$.

For all $i \in N$, let L_i be the *directed tree rooted at* i induced by the tree L. Let $\bar{s}_i(j)$ be the set of successors of player j in the directed tree L_i, and $s_i(j)$ the set of immediate successors of player j.

Definition (Hierarchical Solutions)**.** For all agents $i \in N$, *i-hierarchical solution* $h^i(\cdot) = (h^i_j(\cdot))_{j\in N}$ associates with each $v \in \mathcal{V}$ the payoff vector such that for all $j \in N$,

$$h^i_j(v) = v(\bar{s}_i(j) \cup \{j\}) - \sum_{k\in s_i(j)} v(\bar{s}_i(k) \cup \{k\}).$$

In the environment where coalition formation of a game is also restricted by the same tree network (as in van den Brink 2012), disconnected coalitions T cannot form and so $v(T)$ cannot be realized and the worth of the disconnected coalition should be revised. The greatest realizable payoff for disconnected coalition T is the sum of payoffs for its maximal connected subsets on the tree. Using this, the tree restriction L turns the game v into the *tree-restricted game* v^L such that for all $S \subseteq N$, if S is connected on L, $v^L(S) = v(S)$; if S is disconnected, $v^L(S) = \sum_{T\in P(S)} v(T)$, where $P(S)$ is the set of maximal connected subsets of S on L. Let $\mathcal{V}^L \equiv \{v^L : v \in \mathcal{V}\}$ be the collection of tree-restricted games.

On tree-restricted games, hierarchical solutions and their convex combinations satisfy *null-player property* as well as all the other axioms defined earlier (van den Brink 2012). For the games without tree-restriction, these solutions may violate *null-player property* but they still satisfy *very-null-player property.*

The following facts are useful.

Fact 1. *For all $S, T \subseteq N$, if $S \cap T \neq \emptyset$, then $u^S_T = u^{S\cup T}$.*

Proof. The proof is immediate from the definition. □

Fact 2. *Let $L \subseteq N \times N$ be a tree and $T \subseteq N$. Let u^T be the T-unanimity game. On the TU-game u^T, a player $i \in N$ is a very null player if and only if $i \notin T^{cn}$.*

For a graph (N, L), a subset of players $S \subseteq N$ is called a *subtree* if both of S and $N \setminus S$ are connected.

Fact 3. *Let $S \subseteq N$ be a subtree of a tree L. The for all $T \subseteq N$,*
1. $T \subseteq S$ if and only if $T^{cn} \subseteq S$; and $T \subseteq N \backslash S$ if and only if $T^{cn} \subseteq N \backslash S$.
2. $T \cap S \neq \emptyset$ if and only if $T^{cn} \cap S \neq \emptyset$; and $T \cap (N \backslash S) \neq \emptyset$ if and only if $T^{cn} \cap (N \backslash S) \neq \emptyset$.

Proof. The first equivalence of part 1 is obtained from connectedness of S and the definition of T^{cn}. The second equivalence can be shown using connectedness of $N \backslash S$. Part 2 is equivalent to part 1. □

3 Main Results

We first show that with the exception of the special case that the tree is a line, there does not exist any solution satisfying (*pairwise*) *collusion neutrality*, efficiency, and null-player property.

Theorem 1. *There is a solution on $\mathcal{V}$ satisfying* (pairwise) collusion neutrality*,* efficiency*, and* null-player property *if and only if the tree L is a line.*

Proof. If L is a line, then for any end node i^*, i^*-hierarchical solution $h^i(\cdot)$ satisfies the three axioms. To show this, assume $L \equiv \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}$. Then 1 is an end node and for all $v \in \mathcal{V}$ and all $i \in N$, $h^1_i(v) = v(\{i,i+1,\ldots,n\}) - v(\{i+1,\ldots,n\})$, which equals 0 when i is a null player. Hence 1-hierarchical solution satisfies *null-player property.* Other axioms can be shown easily.

To prove the converse, assume that L is not a line. Then there are distinct four nodes $i,j,k,l \in N$ such that $\{\{i,j\},\{i,k\},\{i,l\}\} \subseteq L$. We prove the non-existence result for the case with $N = \{1,2,3,4\}$ and $L = \{\{1,2\},\{1,3\},\{1,4\}\}$ and skip the straightforward extension of this proof in the general case. We show that given this tree, there is no solution f satisfying the three axioms. For brevity, for all $i,j,k \in N$, we denote $\{i,j\}$ by ij, $\{i,j,k\}$ by ijk. Let $f(u^N) = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. Then, by *collusion neutrality* and Fact 1, $f_{12}(u^{234}) = f_{12}(u^N)$, $f_{13}(u^{234}) = f_{13}(u^N)$, and $f_{14}(u^{234}) = f_{14}(u^N)$. Taking the sum of the three equations and using *efficiency* and *null-player property*, we get $1 = 1 + 2f_1(u^N)$. Hence $\alpha_1 = 0$.

By *collusion neutrality* and Fact 1, $f_{12}(u^{134}) = f_{12}(u^N) = \alpha_1 + \alpha_2$. Since $\alpha_1 = 0$, $f_1(u^{134}) = \alpha_2$. On the other hand, by Fact 1, $(u^{34})_{13} = u^{134} = (u^{34})_{14}$. Thus, since 1 is a null player in u^{34}, $f_{13}(u^{34}) = f_3(u^{34}) = f_{13}(u^{134})$ and $f_{14}(u^{34}) = f_4(u^{34}) = f_{14}(u^{134})$. Taking the sum of the two equations and using *efficiency* and *null-player property*, we get $1 = 1 + f_1(u^{134})$, that is, $f_1(u^{134}) = 0$. Since $f_1(u^{134}) = \alpha_2$, $\alpha_2 = 0$.

Likewise, $(u^{124})_{13} = u^N$and $(u^{24})_{12} = u^{124} = (u^{24})_{14}$ leads to $\alpha_3 = f_1(u^{124}) = 0$. Also, $(u^{123})_{14} = u^N$ and $(u^{23})_{13} = u^{123} = (u^{23})_{12}$ leads to $\alpha_4 = f_1(u^{123}) = 0$.

Therefore, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, which contradicts *efficiency*. □

As is clear from the proof, the non-existence result relies crucially on the decisions over the set of games without tree restrictions, namely the games in $\mathcal{V}\backslash\mathcal{V}^L$. When the domain is restricted to $\mathcal{V}^L$, all affine combinations of the hierarchical solutions satisfy the three axioms (van den Brink 2012). It is also crucial for the non-existence to require zero payoff to all null players, especially those null-players who are not very-null-players. If *null-player property* is replaced with *very-null-player property*, again all affine combinations of the hierarchical solutions satisfy *collusion neutrality* as well as *efficiency* and *very-null-player property*.[1] We next characterize solutions satisfying these three axioms together with *linearity*.

We first show that the total share for a subtree is represented by a weighted sum of the worth of the subtree coalition and the worth of the complement coalition.

Lemma 1. *If a solution f on $\mathcal{V}$ satisfies* efficiency*,* collusion neutrality*,* very-null player property*, and* linearity*, then for all game $v \in \mathcal{V}$ and all subtree $S \subseteq N$,*

$$f_S(v) = (1 - \alpha_S)v(S) + \alpha_S[v(N) - v(N \setminus S)],$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \equiv f(u^N)$ and $\alpha_S \equiv \sum_{i \in S} \alpha_i$.

Proof. Let $v \in \mathcal{V}$ and let $S \subseteq N$ be a subtree. Note that v can be represented by a linear combination of unanimity games: $v = \sum_{T \subseteq N} \lambda_T(v) u^T$. Then, by *linearity*, $f_S(v) = \sum_{T \subseteq N} \lambda_T(v) f_S(u^T)$.

Partition $2^N\backslash\{\emptyset\}$ into $P_1 \equiv \{T : T^{cn} \subseteq S\}$, $P_2 \equiv \{T : T^{cn} \subseteq N\backslash S\}$, $P_3 \equiv \{T : S \subseteq T^{cn}$ and $T^{cn} \nsubseteq S\}$, and $P_4 \equiv \{T : S \nsubseteq T^{cn} \nsubseteq N\backslash S\}$.

Claim 1. For all $T \in P_1$, $f_S(u^T) = 1$.

Proof. Consider $T \in P_1$. Then for all $i \in N\backslash S$, $i \notin T^{cn}$ and so i is a very null player in the game u^T. Hence by *very-null-player property*, $f_i(u^T) = 0$ for all $i \in N\backslash S$. By *efficiency*, this implies $f_S(u^T) = 1$. ⋄

[1]The proof that hierarchical solutions satisfy *very-null-player property* is tedious. It is available upon request.

Claim 2. For all $T \in P_2$, $f_S(u^T) = 0$ and $f_{N\setminus S}(u^T) = 1$.

Proof. Consider $T \in P_2$. Then for all $i \in S$, $i \notin T^{cn}$, and so i is a very null player in u^T. Hence by *very-null-player property* and *efficiency*, $f_S(u^T) = 0$ and $f_{N\setminus S}(u^T) = 1$. $\diamond$

Claim 3. For all $T \in P_3 \cup P_4$, $f_S(u^T) = \alpha_S$.

Proof. We divide the proof into three cases.

Case 1: $T \in P_3$ and T is connected.

Since in this case, $T \supsetneq S$, $u^T_{N\setminus S} = u^N$. This yields, by *collusion neutrality* and connectedness of $N\setminus S$, $f_{N\setminus S}(u^T) = f_{N\setminus S}(u^N)$. Then, by *efficiency*, $f_{N\setminus S}(u^T) = 1 - f_S(u^T)$ and $f_{N\setminus S}(u^N) = 1 - f_S(u^N)$. Therefore, $f_S(u^T) = f_S(u^N) = \alpha_S$.

Case 2: $T \in P_4$ and T is connected.

When S is a singleton, P_4 is empty. By $T \cap S \neq \emptyset$, it holds that $u^T_S = u^{T\cup S}$, which, by *collusion neutrality*, yields $f_S(u^T) = f_S(u^{T\cup S})$. Since $T \cap (N\setminus S) \neq \emptyset$, we have $S \subsetneq T\cup S$, which implies $f_S(u^{T\cup S}) = \alpha_S$ by Case 1. Then it follows that $f_S(u^T) = f_S(u^{T\cup S}) = \alpha_S$.

Case 3: $T \in P_3 \cup P_4$ and T is disconnected.

Since $T \cap S \neq \emptyset$, $u^T_S = u^{T\cup S}$, then by connectedness of S and *collusion neutrality*, $f_S(u^T) = f_S(u^{T\cup S})$. Since $T^{cn} \cup S \in P_3$ and $T^{cn} \cup S$ is connected, then by Case 1, $f_S(u^{T^{cn}\cup S}) = \alpha_S$. Thus we have only to show $f_S(u^{T\cup S}) = f_S(u^{T^{cn}\cup S})$, which leads to $f_S(u^T) = f_S(u^{T\cup S}) = f_S(u^{T^{cn}\cup S}) = \alpha_S$.

Since $(T\cup S)\cup(T^{cn}\setminus S) = T^{cn}\cup S$ and $(T\cup S)\cap(T^{cn}\setminus S) = T\cap(N\setminus S) \neq \emptyset$, then $u^{T\cup S}_{T^{cn}\setminus S} = u^{T^{cn}\cup S}$. By connectedness of $T^{cn} \setminus S$ and *collusion neutrality*,

$$f_{T^{cn}\setminus S}(u^{T\cup S}) = f_{T^{cn}\setminus S}(u^{T^{cn}\cup S}).$$

Since $[T \cup S]^{cn} = T^{cn} \cup S$, then by *efficiency* and *very-null-player property*, $f_{T^{cn}\cup S}(u^{T\cup S}) = f_{T^{cn}\cup S}(u^{T^{cn}\cup S}) = 1$. Since $T^{cn} \cup S = (T^{cn} \setminus S) \cup S$ and $(T^{cn} \setminus S) \cap S = \emptyset$, then $f_{T^{cn}\setminus S}(u^{T\cup S}) = 1 - f_S(u^{T\cup S})$ and $f_{T^{cn}\setminus S}(u^{T^{cn}\cup S}) = 1 - f_S(u^{T^{cn}\cup S})$. Therefore, $f_S(u^{T\cup S}) = f_S(u^{T^{cn}\cup S})$. $\diamond$

Note that $v(S) = \sum_{T\subseteq S} \lambda_T(v) = \sum_{T\in P_1} \lambda_T(v)$ and $v(N\setminus S) = \sum_{T\subseteq N\setminus S} \lambda_T(v) = \sum_{T\in P_2} \lambda_T(v)$. Hence using Claims 1-3,

$$\begin{aligned} f_S(v) &= \sum_{T\in P_1} \lambda_T(v) f_S(u^T) + \sum_{T\in P_2} \lambda_T(v) f_S(u^T) + \sum_{T\in P_3\cup P_4} \lambda_T(v) f_S(u^T) \\ &= v(S) + \sum_{T\in P_3\cup P_4} \lambda_T(v)\alpha_S, \end{aligned}$$

and, since $v(N) = \sum_{T \in P_1} \lambda_T(v) + \sum_{T \in P_2} \lambda_T(v) + \sum_{T \in P_3 \cup P_4} \lambda_T(v) = v(S) + v(N \backslash S) + \sum_{T \in P_3 \cup P_4} \lambda_T(v)$, then $\sum_{T \in P_3 \cup P_4} \lambda_T(v) = v(N) - v(S) - v(N \backslash S)$. Therefore, $f_S(v) = v(S) + \alpha_S[v(N) - v(S) - v(N \setminus S)] = (1 - \alpha_S)\, v(S) + \alpha_S[v(N) - v(N \setminus S)]$. □

By the above lemma, the payoff to every player $i \in N$ is obtained by subtracting the payoffs to subtree coalitions which are the components (maximal connected subsets) of $N \setminus \{i\}$ from the payoff to the grand coalition. We will show, in the proof of our main result, that this payoff coincides with an affine combination of hierarchical outcomes.

Theorem 2. *A solution on $\mathcal{V}$ satisfies* collusion neutrality*,* efficiency*,* very-null-player property*, and* linearity *if and only if it is an affine combination of hierarchical solutions.*

Proof. Let $i \in N$. Let $d(i)$ be the number of components of $N \setminus \{i\}$. Let $D(i) \equiv \{1, \dots, d(i)\}$. Let S_1, S_2, …, $S_{d(i)}$ be the components of $N \backslash \{i\}$, which are all subtree coalitions. Then

$$\alpha_i = 1 - \alpha_{S_1} - \cdots - \alpha_{S_{d(i)}}. \tag{1}$$

By the previous lemma, for all $j = 1, \dots, d(i)$,

$$f_{S_j}(v) = (1 - \alpha_{S_j})v(S_j) + \alpha_{S_j}[v(N) - v(N \setminus S_j)].$$

For all $t, i \in N$, i's t-hierarchical payoff h_i^t is given by: if $t = i$,

$$h_i^i(v) = v(N) - \sum_{j \in D(i)} v(S_j),$$

and, if $t \neq i$ and for some $j = 1, \dots, d(i)$, $t \in S_j$, then

$$h_i^t(v) = v(N \setminus S_j) - \sum_{k \in D(i) \setminus j} v(S_k).$$

Note that i's share in the hierarchical solutions depends only on the subtree coalition to which the root belongs. Then, the payoff to player i is represented by:

$$\begin{aligned} f_i(v) &= f_N(v) - f_{S_1}(v) - \cdots - f_{S_{d(i)}}(v) \\ &= v(N) - \sum_{j \in D(i)} (1 - \alpha_{S_j})v(S_j) - \sum_{j \in D(i)} \alpha_{S_j}[v(N) - v(N \setminus S_j)] \end{aligned} \tag{2}$$

Note that $\sum_{j\in D(i)} \alpha_{S_j} v(S_j) = \sum_{j\in D(i)}(\sum_{k\in D(i)} \alpha_{S_k}) v(S_j) - \sum_{j\in D(i)} \alpha_{S_j} \sum_{k\in D(i)\setminus j} v(S_k)$. Using this, (2) can be rewritten by

$$\begin{aligned} f_i(v) = &\sum_{j\in D(i)} \alpha_{S_j}[v(N\setminus S_j) - \sum_{k\in D(i)\setminus j} v(S_k)] + (1 - \sum_{j\in D(i)} \alpha_{S_j}) v(N) \qquad (3) \\ &+ \sum_{j\in D(i)} \left(\sum_{k\in D(i)\setminus j} \alpha_{S_k} \right) v(S_j) + \sum_{j\in D(i)} \alpha_{S_j} v(S_j) - \sum_{j\in D(i)} v(S_j). \end{aligned}$$

Since $\alpha_{S_j} = \sum_{t\in S_j} \alpha_t$ for each subtree S_j, it holds that $\alpha_{S_j}[v(N\setminus S_j) - \sum_{k\in D(i)\setminus j} v(S_k)] = \sum_{t\in S_j} \alpha_t h_i^t(v, L)$ for $1 \le j \le d(i)$. Then, the first term in the above equation equals the alpha-weighted sum of hierarchical solutions (excluding i): $\sum_{t\neq i} \alpha_t h_i^t(v, L)$. Then using (1), (3) can be written by:

$$\begin{aligned} f_i(v) &= \sum_{t\neq i} \alpha_t h_i^t(v) + \alpha_i[v(N) - \sum_{j\in D(i)} v(S_j)] \\ &= \sum_{t\neq i} \alpha_t h_i^t(v) + \alpha_i h_i^i(v) \\ &= \sum_{t\in N} \alpha_t h_i^t(v). \end{aligned}$$

□

On the domain of tree-restricted games, all hierarchical solutions and their affine combinations satisfy *null-player property* as shown by van den Brink (2012).

Corollary 1. *A solution on $\mathcal{V}^L$ satisfies* collusion neutrality*,* efficiency*,* null-player property*, and* linearity *if and only if it is an affine combination of hierarchical solutions.*

Proof. Note that all tree restricted games $v \in \mathcal{V}^L$ can be represented by a linear combination of the unanimity games of connected coalitions, that is, $v = \sum_{T\in\mathcal{C}(L)} \lambda_T(v) u^T$, where $\mathcal{C}(L)$ is the set of connected subsets of N on L. The rest of the proof is the same as in the proof of Lemma 1 and the proof of Theorem 2. □

The *average tree solution* (Herings et al., 2008) is a special example of using equal weights, namely, $f(v) = \frac{1}{n}\sum_{i\in N} h^i(v)$. Herings et al (2008) provides an axiomatization of the average tree solution. Our main result gives an alternative axiomatization. We only need to add the following minimal equal treatment property.

Uniform Treatment of Uniforms (Ju et al., 2007). $f_1(u^N) = f_2(u^N) = \cdots = f_n(u^N)$.

It follows from Theorem 2 that:

Corollary 2. *A solution on* $\mathcal{V}$ *satisfies* collusion neutrality*,* efficiency*,* very-null-player property*,* linearity*, and* uniform treatment of uniforms *if and only if it is the average tree solution.*

4 Pairwise Neutrality and Non-Bossiness

We now derive alternative characterization results using *pairwise neutrality*. Our results in this section are obtained from the main results and equivalence between *collusion neutrality* and the combination of *pairwise neutrality* and the following axiom of *non-bossiness*, which says that if two linked players cannot change their total payoff through their collusion, the payoffs of the other players should not be changed either.[2]

Non-Bossiness. For all edges $\{i, j\} \in L$, if $f_i(v_{\{i,j\}}) + f_j(v_{\{i,j\}}) = f_i(v) + f_j(v)$, then for all $h \in N \backslash \{i, j\}$, $f_h(v_{\{i,j\}}) = f_h(v)$.

Hierarchical solutions satisfy *non-bossiness*. To see this, for any link i, j with i closer to the root, the total payoff of another player $h \neq i, j$ depends only on the worth of the coalition of h and all successors of h and the worths of coalitions of each immediate successor of h and her successors. Since these coalitions either include both i and j or include none of i and j, then the coalitional worths are not affected by the collusion of i and j.

Lemma 2. Pairwise neutrality*,* non-bossiness*, and* linearity *together imply* collusion neutrality*.*

Proof. Let f be a solution on $\mathcal{V}$ satisfying *pairwise neutrality*, *non-bossiness*, and *linearity*.

First, we show that for all unanimity games u^T and all connected $S \subseteq N$,

$$f_S(u^T_S) = f_S(u^T). \tag{4}$$

[2]Ju (2012) considers a similar axiom in the framework of allocation problems.

Case 1. $T \cap S = \emptyset$ or $S \subseteq T$. In this case, $u^T_S = u^T$and so (4) holds trivially.

Case 2. $T \cap S \neq \emptyset$ and $S \setminus T \neq \emptyset$. Since S is connected, there is $j_1 \in S \setminus T$ that is adjacent to a node $h_1 \in T \cap S \neq \emptyset$. Let $J_1 \equiv \{j_1\}$ and $S_1 \equiv [T \cap S] \cup J_1$. Next, if $S \neq S_1$, there is $j_2 \in S \setminus S_1$ that is adjacent to a node $h_2 \in S_1$. Let $J_2 \equiv \{j_1, j_2\}$ and $S_2 \equiv [T \cap S] \cup J_2$. Continuing this way, elements in $S \setminus T$ can be ordered as $j_1, j_2, \ldots, j_M$ for some M such that for some $h_1, h_2, \ldots, h_M \in S$, we have: for all $m = 0, 1, \ldots, M$, $h_m \in S_{m-1}$ and $h_m j_m \in L$, where $J_0 = \emptyset$ and $S_0 \equiv T \cap S$.

For all $m = 1, \ldots, M$, by *pairwise neutrality*, $f_{h_m j_m}(u^{T \cup J_{m-1}}) = f_{h_m j_m}(u^{T \cup J_m})$. Then by non-bossiness, $f_k(u^{T \cup J_{m-1}}) = f_k(u^{T \cup J_m})$, for all $k \in S \setminus \{h_m, j_m\}$. Hence, for all $m = 1, \ldots, M$, $f_S(u^{T \cup J_{m-1}}) = f_S(u^{T \cup J_m})$, which implies $f_S(u^T) = f_S(u^{T \cup J_1}) = \cdots = f_S(u^{T \cup J_M}) = f_S(u^{T \cup S})$. Note that since $T \cap S \neq \emptyset$, $f_S(u^{T \cup S}) = f_S(u^T_S)$. Therefore we get $f_S(u^T_S) = f_S(u^T)$.

To show that $f_S(v) = f_S(v_S)$ for all $v \in \mathcal{V}$ and all $S \in \mathcal{C}(L)$, we use *linearity* and the fact that all games can be represented by a linear combination of unanimity games. □

It follows from Lemma 2 and Theorem 2 that:

Theorem 3. *The following are equivalent.*
(i) *Solution f on $\mathcal{V}$ satisfies* pairwise neutrality*,* non-bossiness*,* efficiency*,* very-null-player property*, and* linearity.
(ii) *Solution f on $\mathcal{V}$ satisfies* collusion neutrality*,* efficiency*,* very-null-player property*, and* linearity.
(iii) *Solution f is an affine combination of hierarchical solutions.*

Proof. By Lemma 2, (i) implies (ii). Theorem 2 shows the equivalence between (ii) and (iii). Finally, as we have explained earlier, all hierarchical solutions and their affine combinations satisfy the axioms in (i). □

5 Concluding Remarks

A minimal equity condition is that in N-unanimity game u^N where all players take uniform roles, no player should be punished when a player is not punished.

Minimal Equity. There is no pair $i, j \in N$ such that $f_i(u^N) < 0 \leq f_j(u^N)$.

Efficiency and *minimal equity* together imply that for all $i \in N$, $f_i(u^N) \geq 0$ and so all coefficients used in an affine combination of hierarchical solutions in Theorem 2 should be non-negative. Thus adding *minimal equity* to the four axioms in Theorem 2 and Corollary 1, we can characterize the family of *convex combinations* (affine combinations with positive coefficients) of hierarchical solutions.

Collusion neutrality in Theorem 2 cannot be replaced with *pairwise neutrality*. The next example shows that there are other solutions satisfying *pairwise neutrality* and the other three axioms in the theorem.

Example. Let $N = \{1,2,3,4\}$ and $L = \{\{1,2\},\{2,3\},\{3,4\}\}$. For all $T \subseteq N$, the unanimity game for T is denoted by u^T. Define solution f on the set of some of these unanimity games as follows:

$$\begin{aligned} f(u^N) &= (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}) = f(u^{14}) = f(u^{124}) = f(u^{134}), \\ f(u^{123}) = f(u^{234}) &= (0,\frac{1}{2},\frac{1}{2},0) = f(u^{13}) = f(u^{24}), \\ f(u^{23}) &= (0,\frac{1}{2},\frac{1}{2},0), \\ f(u^{12}) &= (0,1,0,0), \\ f(u^{34}) &= (0,0,1,0). \end{aligned}$$

Note that $u^{123}_{34} = u^N$, $u^{12}_{23} = u^{123}$, and $u^{23}_{12} = u^{123}$. For all $v \in \mathcal{V}$, when $v = \sum_{T \subseteq N} \lambda_T(v) u^T$ for some $(\lambda_T(v) \in \mathbb{R})_{T \subseteq N}$, $f(v) \equiv \sum_{T \subseteq N} \lambda_T(v) f(u^T)$. Then it is easy to check that this solution satisfies *pairwise neutrality*, as well as *efficiency*, *very-null-player property*, and *linearity*. However, collusion of $\{2,3,4\}$ at u^{13} reduces the total payoff from $f_{234}(u^{13}) = 1$ to $f_{234}(u^{13}_{234}) = f_{234}(u^N) = 3/4$. Hence the solution violates *collusion neutrality*.

On the domain of tree-restricted games, Mishra and Talman (2010) introduce the following axiom as the key axiom characterizing the average tree solution.

Independence in Unanimity Games, briefly **IUG**. For all $T \subseteq N$ and all $j \in N \setminus T$, if T and $T \cup \{j\}$ are connected, then for all $i \in T$ with $\{i,j\} \notin L$, $f_i(u^T) = f_i(u^{T \cup \{j\}})$.

In the next lemma, we show that *IUG* together with the other axioms in Theorem 2 imply *pairwise neutrality* and *non-bossiness*. Thus, by Lemma 2, *IUG* also implies *collusion neutrality*.

Lemma 3. *Let f be a solution on $\mathcal{V}^L$ satisfying* efficiency*,* null player property*, and* linearity*. Then, if f satisfies* IUG*, then it satisfies* pairwise neutrality *and* non-bossiness.

Proof. Let f be a solution on $\mathcal{V}^L$ satisfying *IUG* as well as *efficiency*, *null player property*, and *linearity*. First, we show that for all connected $T \subseteq N$, all $ij \in L$, and all $k \in N \setminus \{i, j\}$, $f_{ij}(u^T) = f_{ij}(u^T_{ij})$ and $f_k(u^T) = f_k(u^T_{ij})$. If $T \cap \{i, j\} = \emptyset$ or $T \cap \{i, j\} = \{i, j\}$, then $u^T_{ij} = u^T$ and all of equalities trivially hold. Without loss of generality, the remaining case is that $T \cup \{i, j\} = \{i\}$. Since $u^T_{ij} = u^{T \cup \{j\}}$, *IUG* implies $f_k(u^T) = f_k(u^{T \cup \{j\}})$, for all $k \in T \setminus \{i\}$. Combining this with *efficiency* leads to $f_{ij}(u^T) = f_{ij}(u^{T \cup \{j\}}) = f_{ij}(u^T_{ij})$. Also, $f_k(u^T) = f_k(u^{T \cup \{j\}})$ holds for all $k \in N \setminus \{i, j\}$ by *IUG* and *null player property*. Finally, to show that $f_{ij}(v) = f_{ij}(v_{ij})$ and $f_k(v_{ij}) = f_k(v)$ for all $v \in \mathcal{V}$, all $ij \in L$, and all $k \in N \setminus \{i, j\}$, we use *linearity* and the fact that all games can be represented by a linear combination of unanimity games. □

Using this lemma and the earlier results, we get:

Corollary 3. *The following are equivalent.*
(i) *Solution f on $\mathcal{V}^L$ satisfies* IUG*,* efficiency*,* null-player property*, and* linearity.
(ii) *Solution f on $\mathcal{V}^L$ satisfies* pairwise neutrality*,* non-bossiness*,* efficiency*,* null-player property*, and* linearity.
(iii) *Solution f on $\mathcal{V}^L$ satisfies* collusion neutrality*,* efficiency*,* null-player property*, and* linearity.
(iv) *Solution f is an affine combination of hierarchical solutions.*

The main characterization results by Mishra and Talman (2010) is also obtained as a corollary.

Corollary 4. *A solution on $\mathcal{V}^L$ satisfies* IUG *(or* pairwise neutrality *and* non-bossiness*),* efficiency*,* null-player property*,* linearity*, and* uniform treatment of uniforms *if and only if it is the average tree solution.*

Proof. This follows from Corollaries 1, 2, and 3 and Theorem 3, and the fact that on $\mathcal{V}^L$, all hierarchical solutions satisfy *null-player property* as explained earlier. □

van den Brink (2009) establishes a related result to this corollary (with *pairwise neutrality* and *non-bossiness*) imposing an axiom which is similar to the combination of *non-bossiness* and *uniform treatment of uniforms*.

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