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Priority, Solidarity and Egalitarianism

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Abstract

The priority view (Parfit 1997) demands that benefiting people should matter more the worse off these people are in "absolute" terms of their well-being. In the model of allocating resources based on individual capabilities (output functions), this view is well represented by *disability monotonicity* that requires no reduction in the amount of resources allocated to an agent after she becomes more disabled. We provide alternative axiomatic characterizations of the extended egalitarian rules (Moreno-Ternero and Roemer 2006) on the basis of *disability monotonicity* and *agreement* (when there is a change in agents' capabilities or total resources, all agents who remain unchanged should be influenced in the same direction: all unchanged agents get more or all get less or all get the same amount as before).

Keywords: priority, solidarity, egalitarianism, agreement, disability monotonicity
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1 Introduction

In the stylistic framework of resource allocation problems proposed by Moreno-Ternero and Roemer (2006, 2012), individuals are characterized by their capabilities, represented as output functions that transform resources into

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"interpersonally comparable" outputs (e.g., educational achievement, infant mortality, patient survival, success in rescuing victims of a disaster). There is no ex post exchange of outputs and no ex post compensation or transfer, say, via money. Even if ex post compensation or transfer is possible, ethical priority is given on the allocation of resources and outputs, and the key norm under investigation is *distributive justice* in this allocation problem.

Moreno-Ternero and Roemer (2006) build a foundation of extended egalitarian norms using the two ethical principles, "priority" (Parfit 1997; Temkin 1993, 2003) and "solidarity"(Thomson 1983; Chun, 1986). Our goal is to add to this contribution by establishing parallel results in a fixed population model on the basis of a more direct formulation of Parfit's priority view and *agreement* (Moulin 1987a; Thomson 1997, 1999; Chun 1999, 2000). *Agreement* requires that when there occurs a shock changing the output functions of some agents (and none is responsible for it), it should influence the other agents in the same direction, that is, all these unchanged agents get more or all get less or all get the same amounts of resources as before.

Parfit (1997, p.213) proposes, as an alternative to the principle of equality,

> *The Priority View*: Benefiting people matters more the worse off these people are.

It implicates redistributive rectification in wide scope of cases as the teleological egalitarian view does. On the other hand, unlike the teleological egalitarian view, it is not vulnerable to the "leveling down objection". Sen's weak equity axiom (Sen 1973) captures this idea by requiring that a person with disability, or less capability of transforming resources into outputs, should receive more resources. The priority axiom by Moreno-Ternero and Roemer (2006) is much stronger. It requires that even when two persons cannot be ordered in terms of disability (one is disabled relative to the other), no one should get more as well as produce more than the other.

In explaining the difference between the priority and the egalitarian views, Parfit (1997, p.214) remarks that

> Egalitarians are concerned with *relatives*: with how each person's level compares with the level of other people. On the Priority View, we are concerned only with people's absolute levels.

In the same vein, Temkin (2003, p.65) remarks that

> the extent to which improvements in a person's well being affects an outcome's goodness depends solely on their absolute level, and the degree to which their well-being would be improved.

Neither Sen's axiom nor the priority axiom by Moreno-Ternero and Roemer (2006) seems to well accommodate the difference emphasized in the quoted remarks. One way of moving away from "relatives" is to consider how allocation rules respond to a change in a person's disability level. Parfit's priority view will support the idea that when a person becomes more disabled, ceteris paribus, she should not get less resources. She can get more than before not because she is disabled relative to another, but because her disability level increases. This is exactly what our priority axiom, *disability monotonicity* requires. We find that *disability monotonicity* is closely related with Sen's axiom and the priority axiom by Moreno-Ternero and Roemer (2006). In fact, these axioms will be alternatively used, together with *agreement*, to characterize the same family of extended egalitarian rules.

One may criticize that the framework is not appropriate since ex post compensation or transfer may be essential for achieving "efficiency". A modest reaction to this point is that the agents in our model put so much weight on the allocation of resources and outputs that any ex post compensation cannot make substantial differences in their welfare (as for lexicographic preferences ordering over the space of resource, output, and ex post compensation). Another reaction, somewhat provocative to some economists, is that the primary concern for us is a moral evaluation of resource-output allocations; preferences satisfaction, relevant to efficiency, is secondary.

The two well-known forms of egalitarianism are the equalization of resources or the equalization of outcomes. A rich family of extended egalitarian rules in-between these two rules can be formulated based on a variety of index-functions associating with each resource-outcome pair a degree of egalitarian-index. An index-egalitarian rule (Moreno-Ternero and Roemer 2006) allocates resources by equalizing egalitarian indices of all agents. Resource-egalitarian rule utilizes the index function that depends only on resources (outcomes do not count). When outcomes are interpreted as welfare, welfare-egalitarian rule utilizes the index function that depends only on outcomes (resources used to produce the outcomes do not count). These two rules are discussed extensively in the literature, in particular, by Dworkin (1981a, 1981b). A rich spectrum of egalitarianism is provided through a variety of index functions between the two extreme rules.

2 Preliminaries

There is a finite number of agents, each of whom utilizes a resource good to produce an output. A total amount of resources is to be allocated among the agents and individual outputs are interpersonally comparable.

Let $N = \{1, 2, \cdots, n\}$ be the set of agents and assume $n \geq 3$. An individual agent $i \in N$ is characterized by her output function $y_i \colon \mathbb{R}_+ \to \mathbb{R}_+$, which is assumed to be continuous, strictly increasing, unbounded, and $y_i(0) = 0$. Let $\mathcal{Y}^*$ be the set of all such output functions and call it the *universal set of output functions.* For all $i, j \in N$ and all $y_i, y_j \in \mathcal{Y}^*$, y_i is *disabled relative to* y_j if for all $w \geq 0$, $y_i(w) \leq y_j(w)$. An *economy* $e \equiv (y, W)$ is composed of a profile of agents' output functions $y \equiv (y_i)_{i \in N} \in \mathcal{Y}^{*N}$ and the total amount of available resources $W \geq 0$. Let $\mathcal{E}^* \equiv \mathcal{Y}^{*N} \times \mathbb{R}_+$ be the set of all economies, the *universal domain.* A domain $\mathcal{E} \subseteq \mathcal{E}^*$ is a non-empty subset of the universal domain such that for some $\mathcal{Y} \subseteq \mathcal{Y}^*$, $\mathcal{E} = \mathcal{Y}^N \times \mathbb{R}_+$.

Domain $\mathcal{E}$ is a *covering domain* if the graphs of output functions in $\mathcal{Y}$ cover the positive quadrant, that is, for all $(a, b) \in \mathbb{R}^2_{++}$, there is $y_i \in \mathcal{Y}$ such that $y_i(a) = b$. It is *well-ordered* if for all two output functions $y_i, y'_i \in \mathcal{Y}$, y_i is disabled relative to y'_i or y'_i is disabled relative to y_i. It is *rich* if for all $y_i, y'_i \in \mathcal{Y}$ and all $a, b \in \mathbb{R}_+$ with $a < b$ and $y_i(a) < y'_i(b)$, there is $y''_i \in \mathcal{Y}$ such that both y_i and y'_i are disabled relative to y''_i and $y''_i(a) < y'_i(b)$. For example, if the domain is *max-closed*, that is, for all $y_i, y'_i \in \mathcal{Y}$, $\max\{y_i, y'_i\} \in \mathcal{Y}$,[1] then the domain is rich.

An *allocation rule* is a function F that associates with each economy $e = (y, W) \in \mathcal{E}$ a vector of individual shares of W, $F(e) = (F_i(e))_{i \in N} \in \mathbb{R}^n_+$ meeting *resource constraint*, $\sum_{i \in N} F_i(e) = W$.

Here are some useful notation. For all $x = (x_i)_{i \in N}$ and $y = (y_i)_{i \in N}$, we write $x > y$ if $x_i > y_i$ for all $i \in N$; $x \geq y$ if $x_i \geq y_i$ for all $i \in N$. For all $S \subseteq N$, let $x_S \equiv (x_i)_{i \in S}$ and let (x'_i, x_{-i}) be the profile obtained from x by replacing its i-th component with x'_i.

3 Axioms

3.1 Priority Axioms

Moreno-Ternero and Roemer (2006) formalize Parfit's priority principle by requiring that when agent i is offered of less resources than agent j, agent i should produce at least as much as agent j; that is, there should not be any domination of the resource-output pairs across any two agents.

No-Domination. For all $e = (y, W) \in \mathcal{E}$, there is no pair $i, j \in N$ such that $F_i(e) < F_j(e)$ and $y_i(F_i(e)) < y_j(F_j(e))$.

[1]For all $a > 0$, $max\{y_i, y'_i\}(a) \equiv max\{y_i(a), y'_i(a)\}$.

Moreno-Ternero and Roemer (2006) call this axiom *priority.* A milder principle, which is more directly connected to Parfit's principle is to require that when agent i is disabled relatively to agent j, more resources should be offered to agent i than to agent j (referred to as the weak equity axiom by Sen 1973).

(Disability-)Order-Preservation. For all $e = (y, W) \in \mathcal{E}$ and all $i, j \in N$, if agent i is disabled relatively to agent j, then $F_i(e) \geq F_j(e)$.

Note that *disability-order-preservation* does not prevent disabled agent i from producing more than less disabled agent j, that is, disabled agent i may receive so much more than agent j that agent i's output may be higher than agent j's, in which case, *no-domination* is violated. The next axiom requires that disabled agent i should not produce more than agent j.

No-Reversal (in Outputs). For all $e = (y, W) \in \mathcal{E}$ and all $i, j \in N$, if agent i is disabled relatively to agent j, then $y_i(F_i(e)) \leq y_j(F_j(e))$.

Note that each of the three axioms implies *equal treatment of equals*; for all $e = (y, W) \in \mathcal{E}$, if $y_i = y_j$, then $F_i(e) = F_j(e)$. Note also that *no-domination* implies *order-preservation* and *no-reversal,* and conversely, if the domain is well-ordered, the combination of *order-preservation* and *no-reversal* implies *no-domination.*

According to Parfit's priority view, a disabled person should be given more attention not because of her disability relative to another but because her disability level (Parfit 1997, p.214). The above priority axioms are concerned with *relatives* (Parfit 1997) since they all connect "whom to give more" with the relative disability comparison. The next axiom is not. We think that Parfit's priority view, distinct from the relative concern, will support the idea that when a person becomes more disabled, ceteris paribus, her resource should not decrease. The disabled person can get more than before not because she is disabled relative to another, but because her disability level increases.

Disability Monotonicity. For all $e = (y, W) \in \mathcal{E}$, all $i \in N$, and all $y_i, y_i' \in \mathcal{Y}$, if agent i with y_i' is disabled relative to herself with y_i, $F_i(y_i', y_{-i}, W) \geq F_i(y, W)$.

Thus, benefiting agent i matters more the worse off she becomes, as is stated in Parfit's priority view.

The logical relation between the above priority axioms will be discussed in Section 4. These priority principles will be considered in combination with other basic axioms in the literature of fair allocation.

3.2 Solidarity Axioms

The first solidarity axiom pertains to a shock in the output functions of some agents or resources. It requires that any such shock should influence all unchanged agents in the same direction, that is, all get more, all get less or all get the same as before (Moulin, 1987a; Thomson, 1997, 1999; Chun, 1999, 2000).

Agreement. For all $e = (y, W) \in \mathcal{E}$ and $e' = (y', W') \in \mathcal{E}$, and all $M \subseteq N$, if $y_M = y'_M$, then $F_M(e) = F_M(e')$ or $F_M(e) > F_M(e')$ or $F_M(e) < F_M(e')$.

An implication of *agreement* is that whenever a subgroup of agents with their output functions unaffected by a shock receives the same total amount after the shock, the allocation of this total amount should remain the same too, that is, all of them should get the same individual shares as before (Moulin 1987b; Chun 1999, 2000, 2006).

Separability. For all $e = (y, W) \in \mathcal{E}$ and $e' = (y', W') \in \mathcal{E}$, and all $M \subseteq N$ such that $y_M = y'_M$, if $\sum_{i \in M} F_i(e) = \sum_{i \in M} F_i(e')$, then $F_M(e) = F_M(e')$.

Another implication of *agreement* is the solidarity that pertains to a resource shock. The axiom says that when a bad or a good resource shock occurs to an economy, all the members should share in the calamity or windfall (Roemer, 1986; Chun and Thomson, 1988).

Resource Monotonicity. For all $e = (y, W) \in \mathcal{E}$ and $e' = (y, W') \in \mathcal{E}$, if $W' > W$, then $F(e') > F(e)$.

Evidently, an implication of *resource monotonicity* is *resource continuity*, that is, for all $y \in \mathcal{Y}$, if a sequence of resources $(W^n : n \in \mathbb{N})$ converges to W, then $(F(y, W^n) : n \in \mathbb{N})$ converges to $F(y, W)$.

4 Main results

We first show that *agreement* is equivalent to the combination of *separability* and *resource monotonicity.*

Proposition 1. *A rule satisfies agreement if and only if it satisfies separability and resource monotonicity.*

Proof. We skip the evident proof that *agreement* implies *separability* and *resource monotonicity.* To prove the converse, let F be a rule satisfying *separability* and *resource monotonicity.* Let $e = (y, W) \in \mathcal{E}$ and $e' = (y', W') \in \mathcal{E}$, and $M \subseteq N$ be such that $y_M = y'_M$. We show that $F_M(e) = F_M(e')$ or

$F_M(e) > F_M(e')$ or $F_M(e) < F_M(e')$. Without loss of generality, assume that $\sum_{i\in M} F_i(e) \geq \sum_{i\in M} F_i(e')$.

If $\sum_{i\in M} F_i(e) = \sum_{i\in M} F_i(e')$, then $F_M(e) = F_M(e')$ by *separability.*

Now consider the case $\sum_{i\in M} F_i(e) > \sum_{i\in M} F_i(e')$. By *resource continuity,* there is W^* such that $\sum_{i\in M} F_i(y, W^*) = \sum_{i\in M} F_i(e')$. Then $W^* > W$ or $W^* < W$. By *resource monotonicity,* $W^* < W$. By *separability,* $F_M(y, W^*) = F_M(e')$. By *resource monotonicity,* $F_M(y, W^*) < F_M(e)$. Therefore, $F_M(e') < F_M(e)$. □

Remark 1. Moreno-Ternero and Roemer (2006) show that *solidarity* is equivalent to the combination of *consistency* and *resource monotonicity.* Proposition 1 is in parallel with their result.

Proposition 2. *Given a rich domain, if a rule satisfies no-reversal, disability monotonicity, and agreement, then it satisfies no-domination.*

Proof. Let $\mathcal{E} \equiv \mathcal{Y}^N \times \mathbb{R}_+$ be a max-closed domain (or any rich domain). Let F be a rule satisfying *agreement, no-reversal,* and *disability monotonicity.*

Step 1. For all $e = (y, W) \in \mathcal{E}$, all $i \in N$, and all $y'_i \leq y_i$, $F_i(y'_i, y_{-i}, W) \geq F_i(y, W)$ and for all $j \neq i$, $F_j(y'_i, y_{-i}, W) \leq F_j(y, W)$.
Let $e = (y, W) \in \mathcal{E}$, $i \in N$, and y'_i be such that $y'_i \leq y_i$. Let $x \equiv F(y, W)$ and $x' \equiv F(y'_i, y_{-i}, W)$. By *disability monotonicity,* $x_i \leq x'_i$. If $x_i = x'_i$, then by *agreement* and *resource constraint,* $x_{-i} = x'_{-i}$. Likewise, if $x_i < x'_i$, $x_{-i} > x'_{-i}$.

Step 2. F satisfies *no-domination.*
Suppose by contradiction that for some $e \equiv (y, W)$ and $i, j \in N$, $x_i < x_j$ and $y_i(x_i) < y_j(x_j)$ where $x \equiv F(e)$. Let $y'_i \in \mathcal{Y}$ be such that $\max\{y_i, y_j\} \leq y'_i$ and $y'_i(x_i) < y_j(x_j)$. Existence of such y'_i is guaranteed by the domain richness. Let $e' \equiv ((y'_i, y_{-i}), W)$ and $x' \equiv F(e')$. Since y_i is disabled relative to y'_i, by Step 1, $x'_i \leq x_i$ and $x_j \leq x'_j$. Then $y'_i(x'_i) \leq y'_i(x_i) < y_j(x_j) \leq y_j(x'_j)$. Hence $y'_i(x'_i) < y_j(x'_j)$, which contradicts *no-reversal* at e'. □

We next define a family of rules that satisfy all *priority* axioms and *agreement.* Let Φ be the class of all functions $\varphi : \mathbb{R}^2_{++} \cup (0,0) \longrightarrow \mathbb{R}_+$, continuous on its domain and non-decreasing, such that $\inf\{\varphi(x, y)\} = \varphi(0, 0) = 0$ and, for all $(x, y) > (z, t), \varphi(x, y) > \varphi(z, t)$. Let φ be a function in the class Φ. For all $i \in I$, define the function $\psi_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ that determines agent i's φ-value as a function of i's wealth, i.e., $\psi_i(w) = \varphi(w, y_i(w))$ for all $w \in \mathbb{R}_+$. Then we can define the corresponding index-egalitarian rule (Moreno-Ternero and Roemer 2006) as the rule that equalizes the φ-value across agents.

Index-Egalitarian Rule E^φ: For all $e = (y, W) \in \mathcal{E}$ and all $i \in N$, $E_i^\varphi(e) = \psi_i^{-1}(\lambda)$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \psi_i^{-1}(\lambda) = W$.

Note that for all $i \in N$, ψ_i^{-1} is a continuous, strictly increasing, and unbounded function that satisfies $\psi_i^{-1}(0) = 0$. All the rules within the family $E^\varphi(e)$ satisfy *no-domination* and *agreement.* Moreover, they are the only rules satisfying the two axioms simultaneously.

Theorem 1. *Given a covering domain, a rule satisfies no-domination and agreement if and only if it is index-egalitarian. When the covering domain is well-ordered, a rule satisfies order-preservation, no-reversal, and agreement if and only if it is index-egalitarian.*

The proof is provided in the appendix. We next establish an alternative characterization replacing *no-domination* with the combination of *no-reversal* and *disability monotonicity.*

Theorem 2. *Given a rich covering domain, a rule satisfies no-reversal, disability monotonicity, and agreement if and only if it is index-egalitarian.*

Proof. The "only-if" part follows from Theorem 1 and Proposition 2. We only have to prove that all index-egalitarian rules satisfy *disability monotonicity.* Let F be an index-egalitarian rule represented by φ. Let $(y, W) \in \mathcal{E}$, $i \in N$, and $y'_i \in \mathcal{Y}$ be such that $y'_i \leq y_i$. Let $y' \equiv (y'_i, y_{-i})$, $x \equiv F(y, W)$ and $x' \equiv F(y', W)$. Then there exist $\lambda, \lambda' \geq 0$ such that for all $h \in N$, $\varphi(x_h, y_h(x_h)) = \lambda$ and $\varphi(x'_h, y'_h(x'_h)) = \lambda'$. First, suppose that $\lambda' > \lambda$. Then for all $h \in N \backslash \{i\}$, $\varphi(x_h, y_h(x_h)) < \varphi(x'_h, y_h(x'_h))$, which implies that $x_h < x'_h$. In the case of i, $\varphi(x_i, y_i(x_i)) < \varphi(x'_i, y'_i(x'_i)) \leq \varphi(x'_i, y_i(x'_i))$, which implies that $x_i < x'_i$. Altogether, $W = \sum_{h \in N} x_h < \sum_{h \in N} x'_h = W$, a contradiction. Therefore, $\lambda' \leq \lambda$. Then for all $h \neq i$, $\varphi(x_h, y_h(x_h)) \geq \varphi(x'_h, y_h(x'_h))$, which implies that $x_h \geq x'_h$. And by *resource constraint,* $x_i \leq x'_i$, as required by *disability monotonicity.* □

It follows from Proposition 1 and Theorems 1 and 2 that:

Corollary 1. *Given a covering domain, a rule satisfies no-domination, separability, and resource monotonicity if and only if it is index-egalitarian. When the covering domain is rich, a rule satisfies no-reversal, disability monotonicity, separability, and resource monotonicity if and only if it is index-egalitarian.*

Appendix: Proof of Theorem 1

We prove Theorem 1 through an adaptation of the proof used by Moreno-Ternero and Roemer (2006). Their solidarity axiom in the variable population model implies an invariance property in the reduced-population-problem, known as "consistency", which plays an essential role in their proof. We cannot utilize the same proof since we consider a different solidarity axiom formulated for the fixed population model.

Fix $\tilde{y}_1 \in \mathcal{Y}$. Given a rule F, for all $\alpha \in R_+$, let $E(\alpha)$ be the set of economies where an agent with $\tilde{y}_1$ exists and any agent with $\tilde{y}_1$ receives α, that is, $\mathcal{E}(\alpha) \equiv \{e \in \mathcal{E} :$ for some $i \in N$, $y_i = \tilde{y}_1$ and for all $j \in N$ with $y_j = \tilde{y}_1$, $F_j(e) = \alpha\}$. Let $C(\alpha)$ be the set of all resource-outcome pairs in all economies in $\mathcal{E}(\alpha)$, that is, $C(\alpha) \equiv \{(a, b) \in \mathbb{R}^2_+ :$ there is $e \in \mathcal{E}(\alpha)$ such that for some $j \in N$, $F_j(e) = a$ and $y_j(a) = b\}$.

Lemma 1. *If F satisfies no-domination and resource continuity, then for all $y \in \mathcal{Y}$, all $M \subset N$, and all $\alpha \in \mathbb{R}_+$, there exists $W^* \in \mathbb{R}_+$ such that $\sum_{i \in M} F_i(y, W^*) = \alpha$.*

Proof. Let $y \in \mathcal{Y}^N$, $M \subseteq N$ and $\alpha \geq 0$. Let $W_1 \in \mathbb{R}_+$ be such that $W_1 < \alpha$. Since $\sum_{i \in N} F_i(y, W_1) = W_1$ and for all $i \in N$, $F_i(e) \geq 0$, then $\sum_{i \in M} F_i(y, W_1) < \alpha$.

We next show that there is $W_2 \geq 0$ such that $\sum_{i \in M} F_i(y, W_2) > \alpha$. Consider a sequence $(W^n \colon n \in \mathbb{N})$ such that $\lim_{n \to \infty} W^n = \infty$. Since for all n, $\sum_{i \in N} F_i(y, W^n) = W^n$, there is $j \in N$ such that $(F_j(y, W^n) : n \in \mathbb{N})$ is an unbounded sequence. Then, since $y_j(\cdot)$ is an unbounded function, $(y_j(F_j(y, W^n)) : n \in \mathbb{N})$ is also an unbounded sequence.

We show that there is $\bar{n}$ such that $\sum_{i \in M} F_i(y, W^{\bar{n}}) > \alpha$. Suppose by contradiction that for all $n \in \mathbb{N}$, $\sum_{i \in M} F_i(y, W^n) \leq \alpha$. Since both $(F_j(y, W^n) : n \in \mathbb{N})$ and $(y_j(F_j(y, W^n)) : n \in \mathbb{N})$ are unbounded, there is n such that $\sum_{i \in M} F_i(y, W^n) \leq \alpha < F_j(y, W^n)$ and for all $i \in M$, $y_i(\alpha) < y_j(F_j(y, W^n))$. Hence for such n, for all $i \in M$, $F_i(y, W^n) \leq \alpha < F_j(y, W^n)$ and $y_i(F_i(y, W^n)) \leq y_i(\alpha) < y_j(F_j(y, W^n))$, which contradicts *no-domination.*

Now let $W_2 \equiv W^{\bar{n}}$. Then $\sum_{i \in M} F_i(y, W_2) > \alpha$. Since $\sum_{i \in M} F_i(y, W_1) < \alpha < \sum_{i \in M} F_i(y, W_2)$, by *resource continuity,* there is $W^* \in \mathbb{R}_+$ such that $\sum_{i \in M} F_i(y, W^*) = \alpha$. □

Lemma 2. *Assume that F satisfies no-domination and agreement. For all $e \equiv (y, W)$ and all three distinct $i, j, k \in N$, there is $e' \equiv (y', W')$ such that $y'_i = y_i$, $y'_j = y'_k = y_j$, and $F_i(e') = F_i(e)$ and $F_j(e') = F_k(e') = F_j(e)$.*

Proof. Let $e \equiv (y, W)$ and i, j, k are distinct. Let y' be such that $y'_i = y_i$, $y'_j = y'_k = y_j$. By Lemma 1, there is W' such that $F_i(e') + F_j(e') = F_i(e) + F_j(e)$, where $e' \equiv (y', W')$. By *separability* (implied by *agreement*), $F_i(e') = F_i(e)$ and $F_j(e') = F_j(e)$. Since $y'_j = y'_k = y_j$, then by *no-domination*, $F_k(e') = F_j(e)$. □

We show that for all $\alpha \geq 0$, $C(\alpha)$ is downward sloping.

Lemma 3. *If F satisfies no-domination and agreement, then $C(\alpha)$ is downward sloping, that is, for all $(a, b), (a', b') \in C(\alpha)$ with $a < a'$, we have $b \geq b'$.*

Proof. Assume that F satisfies *no-domination* and *agreement.* To prove that $C(\alpha)$ is downward sloping, suppose, to the contrary, that for some $(a, b), (a', b') \in C(\alpha)$, $a < a'$ and $b < b'$. By definition of $C(\alpha)$, there exist $e = (y, W) \in \mathcal{E}(\alpha)$ and $e' = (y', W') \in \mathcal{E}(\alpha)$ such that for some $i, j \in N$ such that $(a, b) = (F_i(e), y_i(F_i(e)))$ and $(a', b') = (F_j(e'), y'_j(F_j(e')))$. By Lemma 2, we may let $y_1 = \tilde{y}_1 = y'_1$ and assume that $1, i, j$ are three distinct agents. Note that $F_1(e) = F_1(e') = \alpha$. Let $\hat{y}$ be such that $\hat{y}_{\{1,i,j\}} = y'_{\{1,i,j\}}$ and $\hat{y}_{N\setminus\{1,i,j\}} = y_{N\setminus\{1,i,j\}}$. By Lemma 1, there is $\hat{W}$ such that $F_1(\hat{e}) + F_i(\hat{e}) + F_j(\hat{e}) = F_1(e') + F_i(e') + F_j(e')$, where $\hat{e} \equiv (\hat{y}, \hat{W})$. By *separability* (implied by *agreement*), $F_{\{1,i,j\}}(\hat{e}) = F_{\{1,i,j\}}(e')$.

Let y'' such that $y''_i = y_i$, $y''_j = y'_j$, $y''_1 = \tilde{y}_1$ and for all $h \neq i, j, 1$, $y''_h = y_h$. By Lemma 1, there is $W'' \geq 0$ such that

$$F_1(e'') + F_i(e'') + F_j(e'') = \alpha + a + a', \tag{1}$$

where $e'' \equiv (y'', W'')$. Suppose $F_1(e'') > \alpha$. By applying *agreement* to e and e'', we get $F_i(e'') > a$. Likewise, by applying *agreement* to $\hat{e}$ and e'', we get $F_j(e'') > a'$. Altogether, $F_1(e'') + F_i(e'') + F_j(e'') > \alpha + a + a'$, contradicting (1). Therefore $F_1(e'') \leq \alpha$. Similarly, we can show $F_1(e'') \geq \alpha$. Hence $F_1(e'') = \alpha$.

Then by *agreement,* $F_i(e'') = a$ and $F_j(e'') = a'$. Finally, since $(a, b) = (F_i(e''), y''_i(e'')) < (F_j(e''), y''_j(e'')) = (a', b')$, in violation of *no-domination* at e''. □

Lemma 4. *$\{C(\alpha) : \alpha \in \mathbb{R}_+\}$ is a collection of disjoint sets.*

Proof. Let $\alpha_1 > \alpha_2$. Suppose that $(a, b) \in C(\alpha_1) \cap C(\alpha_2)$. Then there exist $e^1 = (y, W^1)$ and $i \in N$ such that $y_1 = \tilde{y}$, $F_1(e) = \alpha_1$, and $(F_i(e^1), y_i(F_i(e^1))) = (a, b)$. By Lemma 1, there is W^2 such that $F_1(y, W^2) = \alpha_2$. Let $e^2 \equiv (y, W^2)$. By *resource monotonicity,* $F_i(e^1) = a > F_i(e^2)$, and so $y_i(F_i(e^1)) = b > y_i(F_i(e^2))$. Since $(a, b) \in C(\alpha_2)$ and $(F_i(e^2), y_i(F_i(e^2))) \in C(\alpha_2)$, $C(\alpha_2)$ is not downward sloping, contradicting the conclusion of Lemma 3. □

The next lemma says, by varying $\alpha \geq 0$, $C(\alpha)$'s can cover the positive quadrant.

Lemma 5. *For all* $(a, b) \in \mathbb{R}^2_{++} \cup \{(0, 0)\}$*, there is a unique* $\alpha \geq 0$ *such that* $(a, b) \in C(\alpha)$*.*

Proof. Let $(a, b) \in \mathbb{R}^2_{++} \cup \{(0, 0)\}$. Since $\mathcal{Y}$ covers the positive quadrant, there exist $y \in \mathcal{Y}^N$ and $i \in N \backslash \{1\}$ such that $y_i(a) = b$ and $y_1(\cdot) = \tilde{y}_1(\cdot)$. By Lemma 1, there exists $W \in \mathbb{R}_+$ such that $F_i(y, W) = a$. By letting $\alpha \equiv F_1(y, W)$, we get $(a, b) \in C(\alpha)$. Finally, the uniqueness of α is implied by Lemma 4. □

The next lemma says that if $\alpha_1 > \alpha_2$, then $C(\alpha_1)$ lies above $C(\alpha_2)$.

Lemma 6. *If* $\alpha_1 > \alpha_2$*, then* (i) *for all* $(a, b) \in C(\alpha_2)$ *there exists* $(a', b') \in C(\alpha_1)$ *such that* $(a, b) < (a', b')$*, and* (ii) *there is no* $(a'', b'') \in C(\alpha_2)$ *and* $(a, b) \in C(\alpha_1)$ *such that* $(a'', b'') > (a, b)$*.*

Proof. Fix $\alpha_1 > \alpha_2$. To prove (i), let $(a, b) \in C(\alpha_2)$. Let $e = (y, W) \in \mathcal{E}(\alpha_2)$ and $i \in N$ be such that $y_1 = \tilde{y}_1$, $F_1(e) = \alpha_2$, and $(F_i(e), y_i(F_i(e))) = (a, b)$. By Lemma 1, there is W' such that $F_1(y, W') = \alpha_1$. Since $\alpha_1 > \alpha_2$, by *agreement,* $F_i(y, W') > F_i(y, W') = a$. Thus by letting $a' \equiv F_i(y, W')$ and $b' \equiv y_i(a')$, we get $(a, b) < (a', b') \in C(\alpha_1)$.

To prove (ii), suppose by contradiction that there exist $(a, b) \in C(\alpha_1)$ and $(a'', b'') \in C(\alpha_2)$ such that $(a'', b'') > (a, b)$. By (i), there is $(a^*, b^*) \in C(\alpha_1)$ such that $(a^*, b^*) > (a'', b'')$. Therefore, $(a^*, b^*) > (a, b)$, which contradicts that $C(\alpha_1)$ is downward sloping. □

The next lemma can be established from the above lemmas as in Moreno-Ternero and Roemer (2006).

Lemma 7. *Given a rule satisfying no-domination and resource monotonicity, define* $\varphi : \mathbb{R}^2_{++} \cup \{(0, 0)\} \to \mathbb{R}_+$ *such that for all* $(a, b) \in \mathbb{R}^2_{++}$*,* $\varphi(a, b) \equiv \alpha$*, where* $\alpha \in \mathbb{R}_+$ *is such that* $(a, b) \in C(\alpha)$*. Then* φ *is well-defined and is continuous, non-decreasing,* $\inf\{\varphi(a, b) : (a, b) \in \mathbb{R}^2_{++}\} = \varphi(0, 0) = 0$*, and for all* $(a, b), (a', b')$ *with* $a < a'$ *and* $b < b'$*,* $\varphi(a, b) < \varphi(a', b')$*.*

Now we are ready to prove Theorem 1.

Proof of Theorem 1. For all $(a, b) \in \mathbb{R}^2_{++} \cup \{(0, 0)\}$, let $\varphi(a, b) \equiv \alpha$, where α is such that $(a, b) \in C(\alpha)$. By Lemma 5, $\varphi(\cdot)$ is well-defined. By Lemma 7, $\varphi \in \Phi$.

We now show that $F(y, W) = E^\varphi(y, W)$ for all $(y, W) \in \mathcal{E}$. Let $e = (y, W) \in \mathcal{E}$.

If for some $i \in N$, $y_i = \tilde{y}_1$, then by letting $\lambda = F_i(e)$, we have for all $j \in N$, $(F_j(e), y_j(F_j(e))) \in C(\lambda)$. Therefore $\psi_j(F_j(e)) = \varphi(F_j(e), y_j(F_j(e))) = \lambda$ for all j. Since $\sum_{j \in N} F_j(e) = W$, $F(e) = E^\varphi(e)$.

We now consider the case that there is no $i \in N$ with $y_i = \tilde{y}_1$. We will show that there is unique $\alpha \geq 0$ such that $(F_h(e), y_h(F_h(e))) \in C(\alpha)$ for all $h \in N$. Consider $y' \equiv (\tilde{y}_1, y_2, \ldots, y_n)$. By Lemma 1, there is W' such that $\sum_{h \in N \setminus \{1\}} F_h(y', W') = \sum_{h \in N \setminus \{1\}} F_h(e)$. By *separability* (implied by *agreement*), for all $h \in N \setminus \{1\}$, $F_h(y', W') = F_h(e)$. Hence for all $h \in N \setminus \{1\}$, $(F_h(e), y_h(F_h(e))) \in C(\alpha)$. Similarly, we can show that $(F_1(e), y_1(F_1(e))) \in C(\alpha)$. Therefore, for all $h \in N$, $\psi_h(F_h(e)) = \varphi(F_h(e), y_h(F_j(e))) = \alpha$ and $F(e) = E^{\varphi}(e)$. □

References

[1] Chun, Y. (1986), The solidarity axiom for quasi-linear social choice problems, *Social Choice and Welfare* 3, 297-310.

[2] Chun, Y. (1999), Equivalence of axioms for bankruptcy problem, *International Journal of Game Theory* 28, 511-520.

[3] Chun, Y. (2000), Agreement, separability, and other axioms for quasi-linear social choice problems, *Social Choice and Welfare* 17, 507-521.

[4] Chun, Y. (2006), The separability principle in economies with single-peaked preferences, *Social Choice and Welfare* 26, 239-253.

[5] Chun, Y., and W. Thomson (1988), Monotonicity properties of bargaining solutions when applied to economics, *Mathematical Social Sciences* 15, 11-27.

[6] Dworkin, R. (1981a), What is equality? Part 1: Equality of welfare, *Philosophy and Public Affairs* 10, 185-246.

[7] Dworkin, R. (1981b), What is equality? Part 2: Equality of resources, *Philosophy and Public Affairs* 10, 283-345.

[8] Moreno-Ternero, J.D., and J.E. Roemer (2006), Impartiality, priority and solidarity in the theory of justice, *Econometrica* 74, No. 5, 1419-1427.

[9] Moreno-Ternero, J.D., and J.E. Roemer (2012), A common ground for resource and welfare egalitarianism, *Games and Economic Behavior* 75, 832-841.

[10] Moulin, H. (1987a), The pure compensation problem: egalitarianism versus laissez-fairism, *Quarterly Journal of Economics* 102, 769-783.

[11] Moulin, H. (1987b), Equal or proportional division of a surplus, and other methods, *International Journal of Game Theory* 26, 11-25.

[12] Parfit, D. (1997), Equality and priority, *Ratio* X, 202-212.

[13] Roemer J.E. (1986), Equality of resources implies equality of welfare, *Quarterly Journal of Economics* 101, 751-784.

[14] Sen, A.K. (1973), *On Economic Inequality*, Clarendon Press, Oxford.

[15] Temkin, L. (1993), *Inequality*, Oxford University Press.

[16] Temkin, L. (2003), Equality, priority or what?, *Economics and Philosophy* 19, 61-87.

[17] Thomson, W. (1983), Problems of fair division and the egalitarian solution, *Journal of Economic Theory* 31, 211-226.

[18] Thomson, W. (1997), The replacement principle in economies with single-peaked preferences, *Journal of Economic Theory* 76, 145-168.

[19] Thomson, W. (1999), Welfare-domination under preference replacement: a survey and open questions, *Social Choice and Welfare* 16, 373-394.

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