Solving Reduced-form Linear Rational Expectations Models*

Jae Won Lee† Woong Yong Park‡

Abstract

This paper proposes an improvement on popular solution methods for linear rational expectations models (for example, Sims 2002) in terms of computational performance: When a model can be transformed into a reduced form, the QZ decomposition to decouple the dynamics of the stable and unstable block of the model can be replaced with the Schur decomposition. The latter runs faster. The new method is applicable to a wide class of models in the literature. It is especially useful for a large-scale model such as a multisector model and a heterogeneous agent model that are increasingly popular recently. Compared to the method that uses the QZ decomposition, the new method that uses the Schur decomposition reduces computing time by 33.0% for a medium-scale model with 39 equations and 91.3% for a large-scale model with 1,908 equations.

Keywords: solution methods for linear rational expectational models, QZ decomposition, Schur decomposition, multisector model, heterogeneous agent model

1 Introduction

Quantitative macroeconomic analysis using estimated dynamic stochastic general equilibrium (DSGE) models, best exemplified by Smets and Wouters (2007), has become popular in academia and in

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†Department of Economics, University of Virginia.
‡Department of Economics and Institute of Economic Research, Seoul National University.

central banks around the world. One of the key steps of such analysis is to solve the equilibrium equations of a DSGE model that are presented as a system of linear expectational difference equations or a linear rational expectations model. Starting with Blanchard and Kahn (1980), several methods to solve linear rational expectations models have been proposed in the literature. The solution methods that were later developed, such as Uhlig (1999), Klein (2000) and Sims (2002), use the QZ decomposition (or the generalized Schur decomposition) to decouple the stable and unstable block of the system. While the QZ decomposition has general applicability, it comes with computational costs that can grow rapidly as the dimensionality of the model increases or the model has to be solved many times for a Monte Carlo simulation. It can be prohibitively slow for large-scale models such as a multisector model with the production network and a model with heterogeneous agents, both of which are increasingly popular recently in macroeconomic research.

This paper shows that such a curse-of-dimensionality problem can be addressed without a dramatic change to the existing solution methods. Specifically, we propose to replace the QZ decomposition with the Schur decomposition based on an observation that most of the DSGE models in the literature can be transformed into a reduced form. Since the QZ decomposition operates on two coefficient matrices simultaneously while the Schur decomposition operates on a single coefficient matrix, the latter runs faster. This may sound obvious. Using the Schur decomposition, however, incurs an extra computational cost to transform a model into a reduced form. What is less obvious is then whether there is still a computational gain when one takes into account the extra cost, and the gain is significant in practice. Another issue is whether the method is widely applicable and easy to use.

We address these questions and describe how to use the Schur decomposition within the framework of the solution method by Sims (2002), better known by its computer code name `gensys`. We check and report the performance improvement of the new method over the method by Sims (2002) for models of different sizes. The performance gain that obtains by transforming models into a reduced form and using the Schur decomposition, in terms of computing time reduction, ranges from 33.0% for a medium-scale model with 39 equations to 91.3% for a large-scale model with 1,908 equations.

The solution method by Sims (2002) is widely used because of its general and easy applicability. It does not require specifying jump and predetermined variables but explicitly recognizes that expectational errors are naturally attached to equations in rational expectations models. As it uses the QZ decomposition, one can flexibly write a DSGE model and add auxiliary equations without worrying about recasting the model into a reduced form. We later illustrate an example where a DSGE model cannot be transformed into a reduced form so the new method with the Schur decomposition cannot be immediately applied. Such models, however, can be easily modified and recast into a reduced form. Thus our advice is that one use the method with the QZ decomposition when he or she does not want to worry about transforming a DSGE model into a reduced form and does not have to solve the model many times, for example in a development stage, and use the new method with the Schur decomposition when computation time is a concern of the first order, for example in an estimation stage. To this end, we provide two Matlab codes on our websites that implement the original method by Sims (2002) and our method. The first one, `gensys2.m`, is a general version that chooses between the QZ decomposition and the Schur decomposition depending on transformability of a model into a reduced form. The second one, `rfsys.m`, uses the Schur decomposition to solve reduced-form models only. The second code is faster than the first code since some steps of the solution method can be simplified. A user can also choose a faster way to transform a model into a reduced form for his or her own application.

While our proposition is applicable to a wide range of linear rational expectations models, we expect that it will be especially useful to the growing literature that employs multisector models.[1] Most of the existing studies in this literature rely on simulation exercises with fixed parameterization. We observe that the curse of dimensionality is an important reason for the literature to fail to take a full advantage of modern statistical methods, typically showcased in the Bayesian DSGE estimation (see Fernandez-Villaverde, Rubio-Ramirez, and Schorfheide (2016) for a recent survey). Though a small number of papers move further and try to estimate their multisector models, they have to keep the number of sectors manageable and estimate their models

[1] Long and Plosser (1983), Horvath (1998, 2000) and Dupor (1999) are early papers that consider multisector DSGE models. More recent contributions include Foerster et al. (2011), Acemoglu et al. (2012), Carvalho and Gabaix (2013), Atalay (2017), Carvalho and Tahbaz-Salehi (2018), Miranda-Pinto and Young (2019), Carvalho, Lee and Park (2020), and Pasten, Schoenle and Weber (2020).

on moderately disaggregated data (for example, see Bouakez, Cardia and Ruge-Murcia 2009 and Carvalho, Lee and Park 2020). This paper shows that a seemingly small change to the original solution method of linear rational expectations models could open the door to the estimation of multisector models on highly disaggregated data, which in our view is an issue of the first order in the quantitative analysis using multisector models.

A solution method of a linear rational expectations model is not just useful for a linear or log-linearized DSGE model. It is also used as an input when solving DSGE models up to second or higher (for example, see Schmitt-Grohe and Uribe 2004 and Kim, Kim, Schaumburg and Sims 2008). Our method with the Schur decomposition thus can be used when working with a higher-order approximation as well. Ahn, Kaplan, Moll, Winberry and Wolf (2018) use the Schur decomposition to solve a linearized general equilibrium heterogeneous agent model with aggregate shocks. However, the system of equilibrium equations of their continuous-time model lends itself naturally to the Schur decomposition without the need to be transformed to a reduced form as in our method.

2 Method

2.1 Detail

A linear rational expectations model can be written in the following canonical form of Sims (2002)

$$\Gamma_0 y\left(t\right) = \Gamma_1 y\left(t-1\right) + C + \Psi z\left(t\right) + \Pi\eta\left(t\right), \tag{1}$$

for $t = 1, 2, \cdots$, where $y\left(t\right)$ is an $n \times n$ vector of variables, $z\left(t\right)$ is an $m \times 1$ vector of exogenous shocks, $\eta\left(t\right)$ is a $p \times 1$ vector of expectational errors with $E_t\eta\left(t+1\right) = 0$ for all t, and C is an $n \times 1$ vector of constants. Coefficient matrices Γ_0, Γ_1, Ψ, and Π are $n \times n$, $n \times n$, $n \times m$, and $n \times p$, respectively. It is assumed that there is a common bound $\bar{\xi}$ on the maximal growth rate of the variables in y. Typically $\bar{\xi}$ is set slightly above a unity so that unit roots are classified as stable.

Suppose that Γ_0 is invertible and the canonical form (1) is recast into the form

$$y(t) = \tilde{\Gamma}_1 y(t-1) + \tilde{C} + \tilde{\Psi} z(t) + \tilde{\Pi} \eta(t), \tag{2}$$

for $t = 1, 2, \cdots$, where $\tilde{\Gamma}_1 = \Gamma_0^{-1}\Gamma_1$, $\tilde{C} = \Gamma_0^{-1}C$, $\tilde{\Psi} = \Gamma_0^{-1}\Psi$, and $\tilde{\Pi} = \Gamma_0^{-1}\Pi$. We call (2) a *reduced form* of the linear rational expectations model (1) following the simultaneous equations model literature. Once a linear rational expectations model is transformed into a reduced form, the Schur decomposition can be applied in lieu of the QZ decomposition to decouple the stable and unstable block of the system. The former is faster than the latter.

Note that the entries of $\tilde{\Gamma}_1$ of all the models that we consider are real. Then there always exists a Schur decomposition of $\tilde{\Gamma}_1$ such that

$$\tilde{\Gamma}_1 = Q \Omega Q', \tag{3}$$

where Q is unitary with $QQ' = Q'Q = I_n$ and Ω is upper block-triangular or upper quasi-triangular with 1×1 or 2×2 diagonal blocks (Golub and Van Loan 1996). The entries of both Q and Ω are real. The real eigenvalues of $\tilde{\Gamma}_1$ are in 1×1 blocks of the diagonal of Ω while its complex conjugate eigenvalue pairs are given as the eigenvalues of 2×2 blocks on the diagonal of Ω. We organize Q and Ω so that the eigenvalues smaller than $\bar{\xi}$ in modulus appear on the upper left diagonal of Ω and those that are greater than or equal to $\bar{\xi}$ in modulus appear on the lower right diagonal of Ω.

We can now proceed with (3) and decouple the system of equations (2) into the stable and unstable block. After such decoupling, the procedures that follow are exactly identical to those in Sims (2002) once we let $\Gamma_0 = I_n$ and $\Gamma_1 = \tilde{\Gamma}_1$, $C = \tilde{C}$, $\Psi = \tilde{\Psi}$, and $\Pi = \tilde{\Pi}$ in the canonical form (1). Since the QZ decomposition of an identity matrix is trivially the same identity matrix, some expressions of Sims (2002) can be simplified.

Before concluding this subsection, we make two additional points on the Schur decomposition. First, we obtain an extra computational gain by using a real from of the Schur decomposition in

(3) rather than its complex form. Suppose that Ω in (3) is partitioned as follows:

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0_{n_2 \times n_1} & \Omega_{22} \end{bmatrix},$$

where Ω_{11}, Ω_{12}, and Ω_{22} are $n_1 \times n_1$, $n_1 \times n_2$, and $n_2 \times n_2$, respectively, with $n_1 + n_2 = n$. Then Ω_{11} and Ω_{22} are also upper quasi-triangular, and the set of the eigenvalues of Ω is the union of the eigenvalues of Ω_{11} and Ω_{22}. For $s = 1, 2, \cdots$, $(\Omega_{22})^s$ is again upper quasi-triangular, and its eigenvalues are those of Ω_{22} raised to the power of s. Therefore, even if Ω is not strictly upper triangular, the solution method of Sims (2002) works fine. In the complex form of the Schur decomposition, on the other hand, Ω is given as strictly upper triangular, and all the eigenvalues, real and complex, are placed on its diagonal. To highlight the performance gain of using a real form rather than its complex form, we later report the results in both cases.[2]

Second, although one can use the Jordan decomposition rather than the Schur decomposition to decompose $\tilde{\Gamma}_1$, the Schur decomposition is usually preferred in practical applications. The main reason is the difficulty in developing a stable numerical algorithm for the Jordan decomposition, as discussed, for example, in Golub and Van Loan (1996). In the special case where $\tilde{\Gamma}_1$ has no repeated eigenvalues, one can use the standard eigenvalue decomposition as in Blanchard and Kahn (1980) for which a robust numerical algorithm is available. Repeated eigenvalues, however, frequently show up in applications.

2.2 Invertibility of Γ_0

The Schur decomposition speeds up solving linear rational expectations models, but at the expense of generality. Thus a natural question is whether the new method is widely applicable. A necessary

[2] It is straightforward to compute the eigenvalues of $\tilde{\Gamma}_1$ from Ω. The QZ decomposition also has a real form. It is faster to compute the real QZ decomposition than the complex QZ decomposition. However, it requires more work in Matlab to extract the generalized eigenvalues from the upper quasi-triangular matrix of the real QZ decomposition than from the strictly upper triangular matrix of the complex QZ decomposition. So, depending on the size of a model, the extra cost to compute the generalized eigenvalues from the real QZ decomposition could be larger in Matlab than its performance gain compared to the complex QZ decomposition. The Lapack routine `dgges`, which is used by Matlab to compute the real QZ decomposition, originally returns the generalized eigenvalues but Matlab does not return them. Function `schur` of Julia and function `qz` of package QZ of R return the generalized eigenvalues computed by `dgges`.

condition for a linear rational expectations model to be transformed into a reduced form is that all model variables, endogenous and exogenous, appear on the left hand side of (1) at least once. Otherwise Γ_0 would not be invertible. For illustration, let us consider a simple example where the dynamics of inflation π_t is given as

$$\pi_t = \beta E_t \pi_{t+1} + u_t, \tag{4}$$

for $t = 1, 2, \cdots$ where u_t is serially independent with $E_t u_{t+1} = 0$. If $0 < \beta < 1$, (4) has a unique solution $\pi_t = u_t$. Now suppose that we are also interested in two step-ahead expected inflation so we add $\pi_{t+2|t} = E_t \pi_{t+2}$ to (4). The dynamics of π_t does not change so the solution is still $\pi_t = u_t$, which implies that $\pi_{t+2|t} = 0$. The following canonical form of the extended model however cannot be transformed to a reduced form since $\pi_{t+2|t}$ does not show up on the left hand side:

$$\begin{bmatrix} 1 & -\beta & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \pi_t \\ \pi_{t+1|t} \\ \pi_{t+2|t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \pi_{t-1} \\ \pi_{t|t-1} \\ \pi_{t+1|t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_t + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_t^1 \\ \eta_t^2 \end{bmatrix},$$

where $\pi_{t+j|t} = E_t \pi_{t+j}$, $\eta_t^1 = \pi_t - \pi_{t|t-1}$ and $\eta_t^2 = \pi_{t+1|t} - \pi_{t+1|t-1}$.

For most of the DSGE models in the literature, Γ_0 is invertible but there are some exceptions, one of which is the model of Smets and Wouters (2007). It cannot be immediately transformed into a reduced form since the real interest rate in its flexible-price part does not appear on the left hand side of the canonical form as in the simple example above. We argue that our method is still viable as one can examine the system of equations and modify it for invertibility into a reduced form. In the simple example above, we can easily reduce the extended model into the original model and add an equation for two step-ahead expected inflation, which is a function of the solution for inflation, afterwards. In Smets and Wouters (2007), the real interest rate of the flexible-price part can be eliminated from the system and the reduced system can be solved using our method. The solution for the real interest rate in terms of the other endogenous variables can be easily added after the reduced system is solved.

An extra computational cost to invert Γ_0 is incurred when a linear rational expectations model is transformed into a reduced form. But the cost is smaller in many cases than the performance gain of switching from the QZ decomposition to the Schur decomposition. The extra cost is also partially compensated because some matrix inversions during the solution procedure become unnecessary with the Schur decomposition. For some DSGE models in the literature, such as a simple 3-equation New Keynesian model, the medium-scale model by Smets and Wouters (2007) (after the aforementioned treatment), and the multisector model by Carvalho, Lee and Park (2020), Γ_0 can be inverted analytically before solving those models. Since Γ_0 does not have to be inverted numerically every time, we can save some time further.[3]

3 Performance comparison

In this section the solution method with the Schur decomposition is compared to the original method with the QZ decomposition in terms of computing time to solve several DSGE models, from a small scale to a medium scale and to a large scale.[4] For fair comparison, we invert Γ_0 every time and include its time to the computing time of the method with the Schur decomposition. To separately figure out the performance gain of using a real form of the Schur decomposition rather than its complex form, we try both real and complex form of the Schur decomposition and compare them in terms of computing time.

The new method is tested with the following DSGE models. The first model is a simple New Keynesian model that has 3 equations. Then we consider the model of Smets and Wouters (2007) and the model of Carvalho, Lee and Park (2020) that can be considered as a medium-scale model with 39 and 144 equations, respectively. While the model of Smets and Wouters (2007) is a single-sector New Keynesian model, the model of Carvalho, Lee and Park (2020) is a multisector New Keynesian model with 27 sectors and the roundabout production structure. Lastly, the multisector

[3]Typically Γ_0 can be analytically inverted if the steady state of a DSGE model does not appear in the equilibrium equations. Even if the steady state of a DSGE model enters the equilibrium equations, it can be analytically inverted if the steady state of a DSGE model can be solved analytically – as is the case for many multisector models. For a large-scale model with hundreds or thousands of equations, one can use a symbolic mathematics package such as Symbolic Math Toolbox of Matlab to invert Γ_0. However, it is found in simulation exercises that the gain of inverting Γ_0 analytically can be smaller than the cost of numerically evaluating Γ_0^{-1} for complicated DSGE models. Later in Section 3, we compare the performance of the original method and the new method by numerically inverting Γ_0 for consistency across different models.

[4]We make a few changes in the original `gensys` Matlab code written by Chris Sims so that it works as similarly to the code that implements the new method as possible. Some of these changes were taken from the most recent version of `gensys`, which is written in R by Chris Sims.

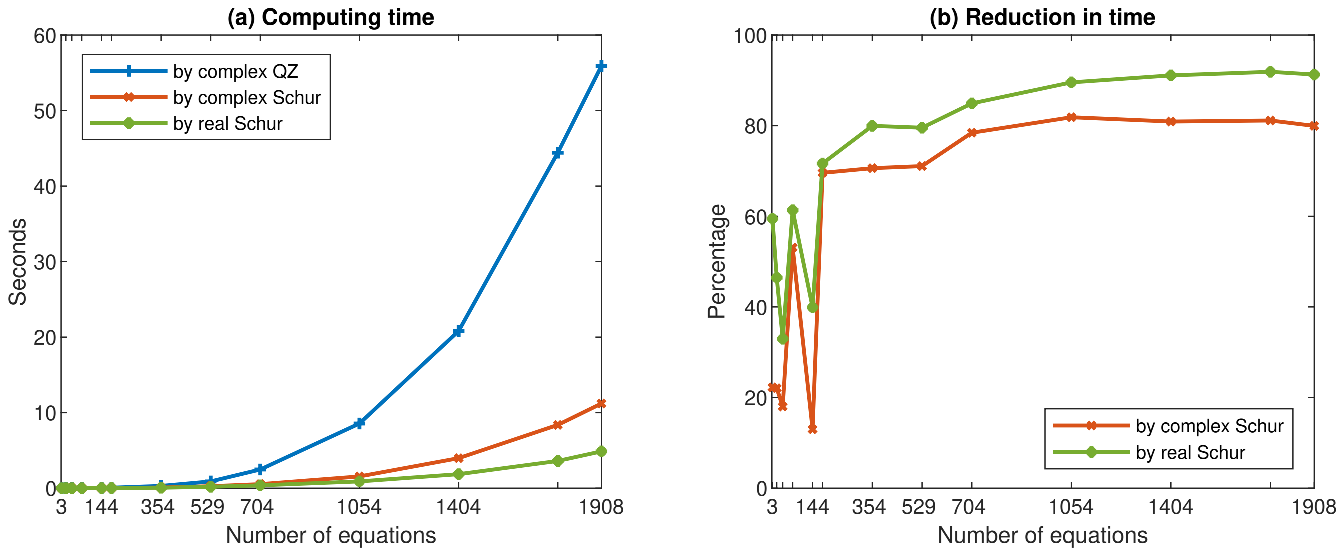

Figure 1: Computing time comparison

Note: The computing time of the solution method with the Schur decomposition includes time to invert Γ_0 numerically. The reduction in time is computed relative to the method using the complex QZ decomposition. The computing time is the average computing time of multiple runs for each model. The exercise is done in Matlab on a computer that is equipped with Intel Xeon CPU E5-2640 v3 (4 physical cores) and 32GB RAM. The details of the comparison exercise are given in the appendix.

New Keynesian model of Lee and Park (2020) is solved with different numbers of sectors. It does not have the roundabout production structure but features a kinked demand function for consumption goods. The largest number of sectors that we consider for the model of Lee and Park (2020) is 272, which is the number of the consumer product categories (entry level items) on which Nakamura and Steinsson (2008) estimate a number of price facts.

The result is reported in Figure 1 and Table 1. First, observe that the computing time to solve a DSGE model rises fast with the dimensionality of the model. It takes almost a minute to solve the model of Lee and Park (2020) with most disaggregate sectors, which is too slow for any realistic simulation and estimation exercises. The computing time is greatly reduced with the Schur decomposition and the reduction in time is bigger with its real form. The relative performance gain is bigger for a larger-scale model. It reaches almost 90% as the number of equations of a model becomes greater than 1,000.

Table 1: Computing time comparison

Models	Number of equations	Computing time (seconds) complex QZ	complex Schur (1)	real Schur (2)	Reduction in time (%) (1)	(2)
Simple NK model	3	0.00018	0.00014	0.00007	22.2	59.5
Smets and Wouters (2007)	39	0.0011	0.0009	0.0007	18.0	33.0
Carvalho, Lee and Park (2020)	144	0.0119	0.0104	0.0072	13.0	39.9
Lee and Park (2020)						
with number of sectors 2	18	0.0005	0.0004	0.0003	22.1	46.5
10	74	0.0060	0.0028	0.0023	53.1	61.3
25	179	0.0637	0.0194	0.0181	69.6	71.6
50	354	0.2963	0.0870	0.0594	70.6	80.0
75	529	0.8937	0.2585	0.1828	71.1	79.5
100	704	2.4731	0.5332	0.3732	78.4	84.9
150	1,054	8.5612	1.5532	0.8941	81.9	89.6
200	1,404	20.8134	3.9729	1.8538	80.9	91.1
250	1,754	44.4219	8.3771	3.6041	81.1	91.9
272	1,908	55.9110	11.2089	4.8719	80.0	91.3

Note: See the note for Figure 1.

4 Conclusion

The original method using the QZ decomposition proposed by Sims (2002) is powerful since one does not have to care about non-invertibility of Γ_0. However, it comes with a cost of relatively slow performance, which can increase rapidly in the dimensionality of a model. The cost is multiplied by millions when one has to solve the model millions of times for a Monte Carlo simulation.

We propose to use the Schur decomposition in lieu of the QZ decomposition after transforming a DSGE model into a reduced form. By switching to the Schur decomposition, we can reduce computing time substantially, up to to 91.3% for a large-scale model with 1,908 equations. Therefore, an efficient strategy would be that one uses the method with the QZ decomposition when he or she does not have to solve a model many times, for example in a development stage, and uses the method with the Schur decomposition when the model has to be solved many times, for example in an estimation stage.

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Appendix

A Details of the performance comparison exercise

We solve the models multiple times and compute the average computing time per run. The number of runs are: 1,000,000 for the simple New Keynesian model and Smets and Wouters (2007) model; 100,000 for Carvalho, Lee and Park (2020); and 100,000 for the 2-sector model, 10,000 for the 10-sector, 25-sector, 50-sector model, 1,000 for the 75-sector, 100-sector, 150-sector, 200-sector, 250-sector, 272-sector of Lee and Park (2020). For larger models, we invert Γ_0 using reusable matrix decompositions in Matlab for faster inversion.

Details of the solution method for Solving Reduced-form Linear Rational Expectations Models *

This note presents the details of the solution method using the Schur decomposition.

A linear rational expectations model can be written in the following canonical form of Sims (2002)

$$\Gamma_0 y(t) = \Gamma_1 y(t-1) + C + \Psi z(t) + \Pi \eta(t), \tag{1}$$

for $t = 1, 2, \cdots$, where $y(t)$ is an $n \times n$ vector of variables, $z(t)$ is an $m \times 1$ vector of exogenous shocks, $\eta(t)$ is a $p \times 1$ vector of expectational errors with $E_t \eta(t+1) = 0$ for all t, and C is an $n \times 1$ vector of constants. Coefficient matrices Γ_0, Γ_1, Ψ, and Π are $n \times n$, $n \times n$, $n \times m$, and $n \times p$, respectively. It is assumed that there is a common bound $\bar{\xi}$ on the maximal growth rate of the variables in y. Typically $\bar{\xi}$ is set slightly above a unity so that unit roots are classified as stable.

If Γ_0 is invertible, the canonical form (1) can be recast into the form

$$y(t) = \tilde{\Gamma}_1 y(t-1) + \tilde{C} + \tilde{\Psi} z(t) + \tilde{\Pi} \eta(t), \tag{2}$$

for $t = 1, 2, \cdots, T$, where $\tilde{\Gamma}_1 = \Gamma_0^{-1}\Gamma_1$, $\tilde{C} = \Gamma_0^{-1} C$, $\tilde{\Psi} = \Gamma_0^{-1}\Psi$, and $\tilde{\Pi} = \Gamma_0^{-1}\Pi$. There always exists a Schur decomposition of $\tilde{\Gamma}_1$ such that

$$\tilde{\Gamma}_1 = Q \Omega Q', \tag{3}$$

where Q is unitary with $QQ' = Q'Q = I_n$ and Ω is upper block-triangular or upper quasi-triangular with 1×1 or 2×2 diagonal blocks. Note that Q' is the conjugate transpose of Q.

Let $w(t) = Q' y(t)$ and pre-multiply both sides of (2) with Q' to obtain

$$w(t) = \Omega w(t-1) + Q' \left[\tilde{C} + \tilde{\Psi} z(t) + \tilde{\Pi} \eta(t) \right]. \tag{4}$$

The diagonal entries of Ω are the eigenvalues of $\tilde{\Gamma}_1$. Let us denote the eigenvalues of $\tilde{\Gamma}_1$ with $\{\omega_{ii}\}_{i=1}^n$. Note that some of the eigenvalues can be repeated. We can rearrange the Schur decomposition (3) so that ω_{ii}'s are placed in the ascending order: $|\omega_{ii}| \leq |\omega_{jj}|$ if $1 \leq i < j \leq n$. Suppose that there are k eigenvalues smaller than $\bar{\xi}$ in the absolute value: $|\omega_{ii}| < \bar{\xi}$ for $1 \leq i \leq k$. We can partition Ω accordingly such that

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix},$$

where Ω_{11} is a $k \times k$ matrix whose diagonal entries are $\{\omega_{ii}\}_{i=1}^k$ and Ω_{22} is an $(n-k) \times (n-k)$ matrix whose

*Jae Won Lee (University of Virginia) and Woong Yong Park (Seoul National University). First version: May 2020. This note follows the description of the solution method by Sims (2002).

diagonal entries are $\{\omega_{ii}\}_{i=k+1}^{n}$. The vectors and matrices in (4) can be partitioned in the same way

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} w_1(t-1) \\ w_2(t-1) \end{bmatrix} + \begin{bmatrix} Q_{1\cdot} \\ Q_{2\cdot} \end{bmatrix} \left[\tilde{C} + \tilde{\Psi} z(t) + \tilde{\Pi}\eta(t)\right], \tag{5}$$

where

$$Q' = \begin{bmatrix} Q_{1\cdot} \\ Q_{2\cdot} \end{bmatrix}.$$

Now note that the lower block of equations in (5) is explosive since the eigenvalues of Ω_{22} are greater than or equal to the bound $\bar{\xi}$. It has a solution that does not explode so long as we solve it forward to make w_2 as a function of future z's. Let us define

$$x_2(t) = Q_{2\cdot}\left[\tilde{C} + \tilde{\Psi} z(t) + \tilde{\Pi}\eta(t)\right].$$

It follows that

$$\begin{aligned} w_2(t) &= \Omega_{22}^{-1} w_2(t+1) - \Omega_{22}^{-1} x_2(t+1) \\ &= \lim_{s\to\infty} \Omega_{22}^{-s} w_2(t+s) - \sum_{s=1}^{\infty} \Omega_{22}^{-s} x_2(t+s) \\ &= -\sum_{s=1}^{\infty} \Omega_{22}^{-s} x_2(t+s), \end{aligned} \tag{6}$$

where in the last equality it is assumed that $\Omega_{22}^{-s} w_2(t+s)$ tends to 0 as s tends to infinity, which will be verified later. This equation implies that the expectational errors should endogenously fluctuate to equate the variable known in period t on the left hand side and a function of the future values on the right hand side. Note that

$$\sum_{s=1}^{\infty} \Omega_{22}^{-s} Q_{2\cdot}\tilde{\Pi}\eta(t+s) = \sum_{s=1}^{\infty} \Omega_{22}^{-s} Q_{2\cdot}\tilde{\Psi}\left[E_t z(t+s) - z(t+s)\right],$$

which leads to

$$Q_{2\cdot}\tilde{\Pi}\eta(t+1) = \sum_{s=1}^{\infty} \Omega_{22}^{-s+1} Q_{2\cdot}\tilde{\Psi}\left[E_t z(t+s) - E_{t+1} z(t+s)\right].$$

For such $\eta(t)$ to exist, it is required that

$$\operatorname{span}\left(Q_{2\cdot}\tilde{\Pi}\right) \supset \operatorname{span}\left(\left\{\Omega_{22}^{-s} Q_{2\cdot}\tilde{\Psi}\right\}_{s=1}^{n-k}\right), \tag{7}$$

according to Sims (2002). If $z(t)$ is serially uncorrelated, the existence condition (7) is reduced to

$$Q_{2\cdot}\tilde{\Pi}\eta(t+1) = -Q_{2\cdot}\tilde{\Psi} z(t+1),$$

which requires that

$$\operatorname{span}\left(Q_{2\cdot}\tilde{\Pi}\right) \supset \operatorname{span}\left(Q_{2\cdot}\tilde{\Psi}\right).$$

Once we confirm that such $\eta(t)$ exists, we can take expectations conditional on the information set available in period t on both sides of (6) to obtain

$$w_2(t) = -\sum_{s=1}^{\infty} \Omega_{22}^{-s} Q_{2\cdot} E_t\left[\tilde{C} + \tilde{\Psi} z(t+s)\right] = (I - \Omega_{22})^{-1} Q_{2\cdot}\tilde{C} - \sum_{s=1}^{\infty} \Omega_{22}^{-s} Q_{2\cdot}\tilde{\Psi} E_t z(t+s). \tag{8}$$

If $z(t)$ is serially uncorrelated, the solution for $w_2(t)$ is simplified to

$$w_2(t) = (I - \Omega_{22})^{-1} Q_{2\cdot}\tilde{C}.$$

For the system to have a unique solution, we should be able to pin down the expectational errors for the stable block using the solution found by solving the unstable block forward. Specifically, there should exist a $k \times (n-k)$ matrix Φ such that

$$Q_{1\cdot}\tilde{\Pi} = \Phi Q_{2\cdot}\tilde{\Pi}.$$

Suppose that such Φ exists. Then (5) leads to

$$\begin{bmatrix} I_k & -\Phi \\ 0 & I_{n-k} \end{bmatrix} w(t) = \begin{bmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \\ 0 & \Omega_{22} \end{bmatrix} w(t-1) + \begin{bmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ Q_{2\cdot} \end{bmatrix} \tilde{C} + \begin{bmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ Q_{2\cdot} \end{bmatrix} \tilde{\Psi} z(t) + \begin{bmatrix} 0 \\ Q_{2\cdot} \end{bmatrix} \tilde{\Pi}\eta(t).$$

Replace the process for $w_2(t)$ with its solution (8) to obtain

$$\begin{aligned} w(t) = {} & \begin{bmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \\ 0 & 0 \end{bmatrix} w(t-1) + \begin{bmatrix} Q_{1\cdot} - \Phi\left[I - (I - \Omega_{22})^{-1}\right] Q_{2\cdot} \\ (I - \Omega_{22})^{-1} Q_{2\cdot} \end{bmatrix} \tilde{C} + \begin{bmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{bmatrix} \tilde{\Psi} z(t) \\ & - \begin{bmatrix} \Phi \\ I \end{bmatrix} \sum_{s=1}^{\infty} \Omega_{22}^{-s} Q_{2\cdot} \tilde{\Psi} E_t z(t+s), \end{aligned}$$

which, by pre-multiplying $Q = \begin{bmatrix} Q_{\cdot 1} & Q_{\cdot 2} \end{bmatrix}$, can be transformed to,

$$\begin{aligned} y(t) = {} & Q_{\cdot 1} \begin{bmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \end{bmatrix} Q' y(t-1) + Q \begin{bmatrix} Q_{1\cdot} - \Phi\left[I - (I - \Omega_{22})^{-1}\right] Q_{2\cdot} \\ (I - \Omega_{22})^{-1} Q_{2\cdot} \end{bmatrix} \tilde{C} \\ & + Q_{\cdot 1}(Q_{1\cdot} - \Phi Q_{2\cdot}) \tilde{\Psi} z(t) - (Q_{\cdot 1}\Phi + Q_{\cdot 2}) \sum_{s=1}^{\infty} \Omega_{22}^{-s} Q_{2\cdot} \tilde{\Psi} E_t z(t+s). \end{aligned}$$

If $z(t)$ is serially uncorrelated, the solution is reduced to

$$y(t) = Q_{\cdot 1} \begin{bmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \end{bmatrix} Q' y(t-1) + Q \begin{bmatrix} Q_{1\cdot} - \Phi\left[I - (I - \Omega_{22})^{-1}\right] Q_{2\cdot} \\ (I - \Omega_{22})^{-1} Q_{2\cdot} \end{bmatrix} \tilde{C} + Q_{\cdot 1}(Q_{1\cdot} - \Phi Q_{2\cdot}) \tilde{\Psi} z(t).$$

If there are no stable roots, that is $k = 0$, $Q = Q_{2\cdot} = Q_{\cdot 2}$ and the solution is simply

$$y(t) = Q_{\cdot 2}(I - \Omega_{22})^{-1} Q_{2\cdot}\tilde{C} - Q_{\cdot 2} \sum_{s=1}^{\infty} \Omega_{22}^{-s} Q_{2\cdot} \tilde{\Psi} E_t z(t+s).$$

If $z(t)$ is serially uncorrelated, the solution is simplified to

$$y(t) = Q_{\cdot 2}(I - \Omega_{22})^{-1} Q_{2\cdot}\tilde{C}.$$

References

[1] Sims, C.A. (2002). "Solving Linear Rational Expectations Models," *Computational Economics*, 20(1-2), 1-20.