The Emergence and Persistence of Oligarchy: A Dynamic Model of Endogenous Political Power*

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Abstract

We study an infinite-horizon multilateral bargaining game in which the status quo policy, players' recognition probabilities, and their voting weights are endogenously determined by the previous bargaining outcome. With players not discounting future payoffs, we show that the long-run equilibrium outcome features the concentration of power by one or two players, depending on the initial bargaining state. If the players' initial shares are relatively equal, they successfully prevent tyranny, but a two-player oligarchy nevertheless emerges and persists. The same results are obtained with payoff discounting, provided that the players' shares are not too small. Our results highlight the importance of the initial power distribution and discounting of future payoffs in the long-run development of power configuration.

Keywords: Dynamic bargaining, Endogenous political power, Endogenous institution, Markov perfect equilibrium, Oligarchy

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1 Introduction

In this paper, we present an infinite-horizon, multiplayer divide-the-dollar game in which the status quo and procedural bargaining power are endogenously determined. In each period, one of the players is selected as a proposer and proposes a division of a dollar. The proposal is implemented if the players possessing more than half of the total voting weights support it; otherwise, the status quo is sustained. Importantly, we assume that the bargaining outcome in a given period determines the status quo policy, the proposer-selection probabilities, and the voting weights in the next period. Simply put, the players who gain more resources in today's bargaining are more likely to be tomorrow's proposer, and their votes would count more heavily than those who gain relatively fewer resources.

This model captures situations in which economic elites have better access to political processes, and thus, the resource distribution is reconsidered under stronger influence by the wealthy. Our model not only applies to the endogenous evolution of institutions (Acemoglu and Robinson, 2000, 2006) but also portrays common phenomena in a democratic society. From Montana's William A. Clark, who bought a seat in the U.S. Senate in the early twentieth century to ever-increasing campaign donations by large firms and billionaires in the present, we have numerous examples of economic elites influencing the political process. Incorporating the idea that "wealth begets power, which begets wealth," (Stiglitz, 2011) this paper thus provides a theoretical framework to examine the long-run consequences of the increasing inequality we now witness.

We characterize the set of symmetric Markov perfect equilibria in stage-undominated voting strategies, with the restriction that the proposer proposes the status quo whenever doing so is optimal. Assuming that players do not discount future payoffs, we show that in any equilibrium, *all wealth (and hence power) is concentrated in the hands of a single player (tyrant) or two players (oligarchs) in the long run*. Once players become a tyrant or oligarch, they deter others' entry into the political arena and perpetuate their rule in all subsequent periods. Moreover, we show that the equilibrium behavior is uniquely characterized in all states except those with relatively equal power.

The long-run equilibrium outcome depends on the initial distribution of wealth and power. If there initially exists an overwhelmingly strong player, he becomes a tyrant once selected as a proposer. If the initial distribution of wealth is relatively equal, however, the players block any other player's attempts to be a tyrant. The players expect that opening the way to tyranny eventually leads to zero future payoffs, and thus, any lucrative short-run benefit is dominated by the long-run cost. Nevertheless, the equilibrium bargaining power is ultimately concentrated in the hands of two players, who form a permanent oligarchy.

While institutional detail is scarce, this paper's results suggest the prevalence of unequal power distribution in political and social institutions. The history of human societies encountered numerous all-powerful rulers for a vast amount of time. Furthermore, our results connect with structural realism in international relations, which argues that the unipolar and bipolar structures are among the world's most stable systems (Waltz, 1964; Wohlforth, 1999).

Next, we discuss the role of two main assumptions in our model. First, we consider the role of discounting by setting up and analyzing a game with discounted payoffs. We show that our model's results extend to this case as long as each player's status quo share is not too small, in which case the players can resist a short-term gain from allowing long-run tyranny. Moreover, we show that the additional constraint vanishes as the discounting factor approaches to one. Second, we discuss a general supermajority voting threshold by analyzing a three-player game. As the voting threshold increases, it becomes more difficult for the proposer to induce a state transition. We show that this effect makes a larger set of states become long-run equilibrium outcomes. This result suggests that more equitable power distribution can be sustained under supermajority rules.

Our paper contributes to the literature on dynamic bargaining with an endogenous status quo (see Eraslan et al. (2020) for a comprehensive survey on this topic). The literature builds on the legislative bargaining framework of Baron and Ferejohn (1989) and analyzes bargaining behavior when the previous bargaining outcome remains in effect until the next agreement.[1] Specifically, our paper belongs to a literature that considers purely distributive policies in a multidimensional space. The literature finds that there exists a Markov perfect equilibrium with a "rotating dictator" feature (Kalandrakis, 2004, 2010) and that almost any outcome is supported as an equilibrium absorbing state (Bowen and Zahran, 2012; Richter, 2014; Anesi and Seidmann, 2015). These papers assume that the bargaining rule (the recognition probabilities and voting weights) remains fixed. In contrast, in our model with an endogenous bargaining rule, every equilibrium features at most two players capturing all resources in the long run.

Several bargaining papers have analyzed the effect of an endogenous procedure. Diermeier et

[1]A nonexhaustive list includes Baron (1996), Fearon (1996), Kalandrakis (2004, 2010), Penn (2009), Diermeier and Fong (2011), Duggan and Kalandrakis (2012), Bowen and Zahran (2012), Bowen et al. (2014), Richter (2014), Anesi and Seidmann (2014, 2015), Baron and Bowen (2015), Dziuda and Loeper (2016), Anesi and Duggan (2018), Baron (2019), Nunnari (2021), and Zapal (2020).

al. (2015, 2016, 2020) consider models in which legislators engage in "procedural voting" in each period. Eguia and Shepsle (2015) find a mechanism whereby the endogenously chosen bargaining rule disproportionately favors more senior legislators. Choate et al. (2020) show that a partisan legislator may be willing to delegate proposal-making authority to a party leader. While these papers endogenize proposer selection rule by either separate voting or a delegation, the present paper assumes that both the proposer-selection probability and voting weights are endogenously determined. Finally, Duggan and Kalandrakis (2012) consider a general model of dynamic legislative bargaining that allows endogenous proposer selection and voting rules. They show that a stationary equilibrium exists in pure strategies and provide a number of regularity properties of equilibria.

Our paper also contributes to the literature on the theory of endogenous institutions (Roberts, 2015; Acemoglu and Robinson, 2000, 2006, 2017; Lagunoff, 2009; Acemoglu et al., 2012, 2015, 2018). Acemoglu et al. (2012) set up a general model of dynamic institutional change and characterize the set of dynamically stable states. Importantly, the intuition underlying our equilibrium shares their "slippery slope" argument (p.1449). While Acemoglu et al. (2012) consider games with deterministic proposer sequences, our model assumes proposers are randomly selected. Stochastic proposer selection, combined with endogenous power, provides the current proposer with more options to enable state transitions, leading to a smaller set of dynamically stable states.[2] One of our main implications—that an initial power distribution crucially affects long-run outcome—is related to Acemoglu and Robinson (2017), who consider a dynamic contest model between the state and civil society.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 characterizes the equilibrium behavior and proves the existence of equilibrium. Section 4 discusses the role of the two main assumptions of our model. Section 5 concludes the paper. All proofs are relegated to the Appendix.

2 Model

Setup A set of players, $I = \{1, \ldots, n\}$ with $n \geq 3$, divide a dollar by time $t = 1, \ldots, \infty$. The bargaining outcome at time t is denoted by $x^t \in X$, where $X = \{x \in \mathbb{R}^n | \sum_{i \in I} x_i = 1 \text{ and } x_i \geq 0\}$ is the set of all feasible divisions of the dollar. The bargaining environment in period t is summarized

[2] Moreover, stochastic proposer selection in our paper results in multiple state transitions in equilibrium, which is not observed in Acemoglu et al. (2012). For detailed argument, see Example 1 and footnote 12.

by $E^t = (x^{t-1}, p^t, w^t)$, where the status quo x^{t-1} is the bargaining outcome carried over from period $t-1$, p^t is the probability vector of proposer selection, which we refer to *recognition probability*, and w^t is the players' voting weights. We assume that p^t and w^t endogenously change over time and are equal to the bargaining outcome in the previous period, i.e., $p^t = w^t = x^{t-1}$. That is, the proposal and voting power of the players are proportional to their resources received in the previous period. Since p^t and w^t are redundant, we denote the environment under the status quo x^{t-1} as $E^t(x^{t-1})$. The status quo in the initial period x^0 is exogenously given.

The timing of the game is as follows. At the beginning of period t, one of the players is selected as a proposer according to the probability vector p^t and makes a proposal $y \in X$. Observing the proposal y, all players simultaneously vote on whether to accept or reject it by a weighted voting rule. If the total voting weights in favor of the proposal are strictly greater than 1/2, the proposal y is adopted as the period-t policy x^t; otherwise, the status quo x^{t-1} becomes x^t. At the end of the period, each player receives a share of the dollar as prescribed by x^t. The bargaining at time $t+1$ begins in the new environment $E^{t+1}(x^t)$, and the same process continues repeatedly.

Strategies Throughout the analysis, we focus on stationary Markov strategies. In stationary Markov strategies, the players ignore the complicated history of the past plays, to the extent that the payoff-relevant parameters are identical at the end of different histories (Maskin and Tirole, 2001). In the current model, the only payoff-relevant parameter is the policy in the previous period, x^{t-1}. Dropping the time index, we denote the *state* s as the policy in the previous period. Note that the set of states is equivalent to the set of feasible bargaining outcomes X; henceforth, we refer to X as either 'the set of states' or 'the set of (bargaining) outcomes' depending on the context.

Let $\mathcal{B}(X)$ be the set of Borel probability measures on X with finite support. player i's *proposal strategy* is a map $\mu_i : X \to \mathcal{B}(X)$.[3] We denote by $\mu_i(y|s)$ the probability that player i proposes y in state s. If $\mu_i(y|s) = 1$ for some y, we write $\mu_i(s) = y$ with a slight abuse of notation. The *acceptance set* $A_i : X \rightrightarrows X$ represents player i's voting strategy, where $A_i(s)$ is the set of proposals he accepts in state s. Let $\sigma_i = (\mu_i, A_i)$ denote player i's Markov strategy.

Since players are ex ante identical except for the initial share, symmetry is a natural assumption for strategies. For any permutation $\phi : I \to I$ and any $x \in X$, let $\hat{\phi} : X \to X$ be a map such that $\hat{\phi}(x) = (x_{\phi(1)}, \dots, x_{\phi(n)})$. A Markov strategy profile σ is symmetric if for any ϕ and x, $\sigma_i(\hat{\phi}(x)) =$

[3]The requirement that μ_i has a finite support is for notational simplicity; dropping the requirement does not affect the results of the paper.

$\sigma_{\phi(i)}(x)$ for each $i \in I$. In symmetric Markov strategies, players are not identified by their names. Two players having identical status quo shares in a given state will propose and vote in the same way, and all other players will treat these two players identically.

Preferences Each player's period-t utility from the outcome x^t equals that player's share of the dollar: $u_i(x^t) = x_i^t$. For a sequence of bargaining outcomes $\{x^t\}_{t=1}^T$, player i's T-period utility is the *average sum* of the period utilities:

$$\frac{\sum_{t=1}^T u_i(x^t)}{T}.$$

Note that the players do not discount payoffs, and thus they do not differentiate between current and future payoffs as long as their sum remains the same. This assumption highlights the role of players' intertemporal incentives in the equilibria of our dynamic bargaining environment. In Section 4.1, we analyze the model with discounted payoffs and compare its results to those of the main model.

A Markov strategy profile σ prescribes how all players behave in every state and thus assigns a probability distribution over the set of all possible sequences of outcomes for $t = 1, \ldots, \infty$ from any state s. Let $\mathcal{L}^s$ be the set of all decisive coalitions in state s: $\mathcal{L}^s = \{L \in 2^I | \sum_{i \in L} s_i > \frac{1}{2}\}$; note that decisive coalitions must have strictly greater than 1/2 of the power. Define the social acceptance set in state s by $A(s) = \bigcup_{L \in \mathcal{L}^s} \bigcap_{i \in L} A_i(s)$. The social acceptance set contains all divisions that will be approved by at least one decisive coalition in state s.

Let $v_i^{\sigma,T}(s)$ be player i's T-period continuation value in state s, in which the bargaining is played for T more periods according to a Markov profile σ. Then, $v_i^{\sigma,T}(s)$ is recursively written as

$$v_i^{\sigma,T}(s) = \sum_{j \in I} s_j \sum_y \mu_j(y|s) \left[\frac{u_i(y) + (T-1)v_i^{\sigma,T-1}(y)}{T} \mathbb{1}_{A(s)}(y) + \frac{u_i(s) + (T-1)v_i^{\sigma,T-1}(s)}{T} \mathbb{1}_{X \setminus A(s)}(y) \right], \tag{1}$$

where $\mathbb{1}$ is an indicator function. Intuitively, player i's T-period continuation value is the average expected utility from period 1 to T *before* the identity of the proposer in the first period is known. Then, each player $j \in I$ is selected with probability s_j and proposes $y \in X$ with probability $\mu_j(y|s)$. If y is in the social acceptance set, player i receives instantaneous utility $u_i(y)$ and expects to receive the continuation value from the new state y for the remaining $T-1$ periods. The sum of $u_i(y)$ and $(T-1)v_i^{\sigma,T-1}(y)$ is averaged over T periods. If player j's proposal is socially unacceptable, the status quo s is implemented and gives player i an instantaneous utility of $u_i(s)$ and a continuation value of $v_i^{\sigma,T-1}(s)$.

Let $U_i^{\sigma,T}(x)$ be player i's T-period expected utility given a Markov profile σ if a policy x is implemented in the current period. Then, $U_i^{\sigma,T}(x)$ is given by

$$U_i^{\sigma,T}(x) \equiv \frac{u_i(x) + (T-1)v_i^{\sigma,T-1}(x)}{T}. \tag{2}$$

Using (2), player i's continuation value $v_i^{\sigma,T}(s)$ can be written as

$$v_i^{\sigma,T}(s) = \sum_{j\in I} s_j \sum_{y} \mu_j(y|s)\left(U_i^{\sigma,T}(y)\mathbb{1}_{A(s)}(y) + U_i^{\sigma,T}(s)\mathbb{1}_{X\setminus A(s)}(y)\right). \tag{3}$$

Hereafter, we omit σ and use the simplified notations of $U_i^T(x)$ and $v_i^T(s)$ whenever doing so does not create confusion.

For the preferences of our infinitely repeated bargaining game without discounting, we apply the *overtaking criterion* of Rubinstein (1979). Formally, for $x, y \in X$, player i prefers x to y given a Markov profile σ ($x \succeq_{i,\sigma} y$) if and only if

$$\liminf_{T\to\infty} T(U_i^{\sigma,T}(x) - U_i^{\sigma,T}(y)) \geq 0.$$

Given any Markov strategy profile σ, a player prefers x to y if x yields a higher expected utility than y when T approaches ∞. Under the overtaking criterion, player i strictly prefers x to y whenever $\lim_{T\to\infty} U_i^{\sigma,T}(x) > \lim_{T\to\infty} U_i^{\sigma,T}(y)$. If $\lim_{T\to\infty} U_i^{\sigma,T}(x) = \lim_{T\to\infty} U_i^{\sigma,T}(y)$, the overtaking criterion takes into account the payoff differences in the finite number of initial periods.

To obtain an intuition, consider the following streams of player i's payoffs from sequences of outcomes x, y, z:

$$x : (\frac{1}{3}, \frac{1}{3}, 0, 0, \ldots) \qquad y : (\frac{1}{3}, 0, 0, 0 \ldots) \qquad z : (0, 0, \frac{1}{3}, \frac{1}{3}, \ldots).$$

Since $U_i^T(x) = \frac{2}{3T}$, $U_i^T(y) = \frac{1}{3T}$ and $U_i^T(z) = \frac{T-2}{3T}$ for any $T \geq 2$, we have $z \succ_i x \succ_i y$ under the overtaking criterion. Comparing x and z, the average payoff over an infinite number of periods from x is zero, but the average payoff from z is $\frac{1}{3}$. Between x and y, these two streams of payoffs both yield zero average payoff when $T \to \infty$, but $\lim_{T\to\infty} T[U_i^T(x) - U_i^T(y)] = \lim_{T\to\infty} T \cdot \frac{1}{3T} = \frac{1}{3} > 0$.[4]

[4]The limit of the means criterion is also frequently employed in infinite-horizon games without discounting. In

Equilibrium Concept Our equilibrium concept is Markov perfection with a few refinement conditions.

Definition 1. A stationary Markov strategy profile $\sigma^* = \{(\mu_i^*, A_i^*)\}_{i=1}^n$ is a *symmetric Markov perfect equilibrium in stage-undominated voting strategies with status quo bias* if, for all $i \in I$, $s \in X$ and a permutation $\phi : I \to I$, the following are satisfied:

(a) *symmetry*: $\sigma_i^*(\hat{\phi}(s)) = \sigma_{\phi(i)}^*(s)$,

(b) *stage-undominated voting*: $y \in A_i^*(s) \Longleftrightarrow y \succeq_{i,\sigma^*} s$,

(c) *optimality*: $\mu_i^*(y|s) > 0 \Longrightarrow y \in \{x \in A^*(s) : x \succeq_{i,\sigma^*} x', \forall x' \in A^*(s)\}$,

(d) *status quo bias*: $s \succsim_{i,\sigma^*} y, \forall y \in A^*(s) \Longrightarrow \mu_i^*(s) = s$.

Henceforth, we refer to the equilibrium defined here simply as an *equilibrium*.

Conditions (a)-(c) are identical to those in Kalandrakis (2004). Condition (a) (*symmetry*) states that the profile is robust to any permutation. Condition (b) (*stage-undominated voting*) requires that the players do not vote for the proposals that would yield lower expected utility than the status quo in equilibrium. Condition (b) eliminates a number of uninteresting equilibria.[5] Additionally, Condition (b) implies that the status quo policy is always accepted unanimously.[6] Condition (c) (*optimality*) requires that players make the optimal proposals in the social acceptance set.[7]

Condition (d) (*status quo bias*) requires that if the proposer's set of optimal proposals includes the status quo, then he proposes the status quo instead of randomly selecting other divisions. This tie-breaking rule can be interpreted as assuming a small cost of forming a new coalition on behalf of the proposer. If a proposer expects no additional benefit from altering the status quo, he has no reason to exert the effort necessary to form a new coalition. While Condition (d) simplifies

the limit of the means criterion, $x \succeq_i^{LM} y \Leftrightarrow \lim_{T\to\infty}[U_i^T(x) - U_i^T(y)]$. In the above example, $x \sim_i^{LM} y$ because $\lim_{T\to\infty} \frac{1}{3T} = 0$. The overtaking criterion is more discriminative among alternatives. With respect to the equilibrium analysis, a player becomes indifferent among all allocations in which his share is more than half under the limit of the means, whereas the overtaking criterion differentiates most of those alternatives.

[5]For example, without Condition (b), we can easily construct an equilibrium in which every player votes for any proposal and the proposer in the first period immediately takes the entire share in most states.

[6]Note that Condition (b) requires that the voters accept the proposal when they are indifferent between the proposal and the status quo. Since the proposal space is continuous, any other tie-breaking rule may lead to an open set of socially acceptable policies, which may result in nonexistence of the optimal proposal.

[7]The assumption that the players make only the proposals in the social acceptance set is introduced to simplify the analysis, but dropping the assumption does not affect the results since the status quo policy is always in the social acceptance set.

the equilibrium analysis, relaxing this Condition does not qualitatively change our equilibrium characterization result.[8]

Partitions of States To present the results more concisely, we partition the state space X into several subspaces according to the level of concentration of voting power.

Definition 2. State s is in the set of
(i) *tyrannical states* X_T if $\exists i \in I$ such that $s_i = 1$;
(ii) *dictatorial states* X_D if $\exists i \in I$ such that $\{i\} \in \mathcal{L}^s$ and $s \notin X_T$;
(iii) *oligarchic states* X_O if $\exists L \in \mathcal{L}^s$ such that $\forall i \in L$, i has a veto[9] and $s \notin (X_T \cup X_D)$;
(iv) *collegial states* X_C if $\bigcap_{L \in \mathcal{L}^s} L \neq \emptyset$ and $s \notin (X_T \cup X_D \cup X_O)$; and
(v) *noncollegial states* X_{NC} if $\bigcap_{L \in \mathcal{L}^s} L = \emptyset$.[10,11]

Figure 1 illustrates the partitions of states in a three-player game. The points in the triangle correspond to particular status quo divisions, where the bottom-left corner is $(1, 0, 0)$, the bottom-right corner is $(0, 1, 0)$ and the center-top corner is $(0, 0, 1)$. These three corners constitute X_T, in which one of the players—whom we call a *tyrant*—has all wealth and power. In dictatorial states, one of the players (the *dictator*) has more than half of the dollar. These are depicted as the shaded areas and solid lines in panel (b). The set of oligarchic states consists of the points at which two of the players (*oligarchs*) divide the dollar equally (panel (c)). Panel (d) illustrates collegial states: There is a single player (*collegium player*) whose status quo share is one half, and there are at least two players (*noncollegium players*) possessing positive status quo shares regardless of the number of players. In noncollegial states (panel (e)), every player's share is less than half, and thus, there is no player who belongs to every decisive coalition.

Finally, we denote by X_{D_i} (X_{C_i}) the set of dictatorial (collegial) divisions in which player i is the dictator (collegium player). Similarly, $X_{D_{-i}}$ ($X_{C_{-i}}$) is the set of states in which player i is a nondictatorial player (noncollegium player).

[8] As we shall see, the only part that the status quo bias has a nontrivial effect is Lemma 2. Without the status quo bias, there exist multiple equilibria with different behavior in the states discussed in Lemma 2, but our result regarding the long-run equilibrium outcome remains the same.

[9] player i has a veto if $x \notin A_i(s)$ implies $x \notin A(s)$.

[10] The term "tyranny" is borrowed from Jordan (2006). The definitions of dictatorial, oligarchic, collegial and noncollegial states are consistent with the definitions of voting rules in the social choice literature except for the slight modification introduced to make them mutually exclusive (Austen-Smith and Banks, 2000).

[11] Note that Definition 2 naturally extends to the case with a general voting threshold $q \neq 1/2$. In Section 4.2, we discuss the equilibrium behavior under a supermajority rule.

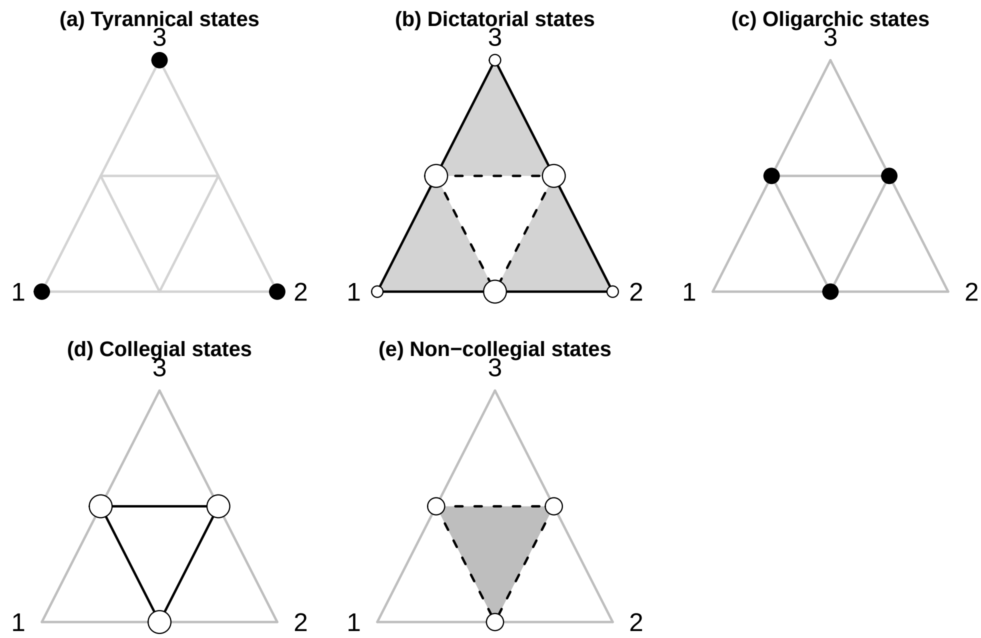


Figure 1: Partitions of states in a three-player game

3 Equilibrium

In this model of endogenous power, the main analytical challenge is that players' preferences over different allocations are affected by future bargaining outcomes. Therefore, the equilibrium behavior of a player in a given state potentially depends on the equilibrium profile in all other states. Our solution is to first analyze states with the most concentrated power structure and then proceed in descending order. As we shall see, in our model, the equilibrium power concentration never decreases over time, and thus our approach significantly simplifies the analysis.

We begin by presenting preliminary results (Lemmas 1-3) that describe the players' equilibrium behavior in each state partition. Theorem 1 then establishes equilibrium existence and summarizes the long-run outcomes and the dynamics in our equilibria.

The first lemma characterizes the unique equilibrium behavior in $s \in X_T \cup X_D \cup X_O$.

Lemma 1 (Tyrannical, dictatorial, and oligarchic states)**.** *In any equilibrium, the following holds:*

(i) For any $s \in X_T$, the tyrant proposes the status quo.

(ii) For any $s \in X_D$, the dictator proposes his tyrannical division, and the other players offer the status quo share to the dictator and take the remainder.

(iii) For any $s \in X_O$, the oligarchs propose the status quo.

Furthermore, all proposals in (i), (ii) and (iii) are accepted by a decisive coalition.

Clearly, a tyrant—whose status quo share is the whole dollar—continues to hold all wealth and power for all future periods. In dictatorial states, the dictator constitutes a decisive coalition by himself. Therefore, he proposes to take the entire dollar and single-handedly passes the tyrannical division. Other players reject the proposal but cannot block the transition to tyranny. If selected to propose, non-dictators make proposals that guarantee the status quo share to the dictator and assign the remainder to themselves. Therefore, there remain at most two players with a positive share after a single round of bargaining in a dictatorial state. The dictator does not accept any shares smaller than the status quo: Protecting his share is important not only for today's wealth but also for determining how rapidly he becomes a tyrant in the future.

For oligarchic states, there are two players who equally share the dollar. These oligarchs together have all proposal power, and each of them has a veto. Neither of the oligarchs can pass a dictatorial division because the other oligarch opposes the plan. Additionally, they do not distribute any share to the third member: Doing so would decrease either of the oligarchs' expected utility. Therefore, both oligarchs propose the status quo, and the oligarchy remains for all future periods.

Given that the equilibrium behavior is uniquely determined, we can derive the equilibrium payoff in each state. It is straightforward that for any T, $U_i^T(s) = 1$ for a tyrant in any $s \in X_T$ and $U_i^T(s) = 1/2$ for an oligarch in any $s \in X_O$. For a dictatorial state, fix a state $s \in X_D$ in which player d's share is greater than $1/2$. Given the equilibrium proposals in item (ii) of Lemma 1, any nondictatorial player $i \neq d$ receives a nonzero payoff only when he is selected as a proposer, in which case he offers $(1 - s_d)$. Furthermore, once another player is selected to propose, player i never becomes a proposer in future periods. Therefore, player i's T-period expected utility in state s is

$$U_i^T(s) = \frac{s_1 + s_1 \sum_{t=1}^{T-1} (1 - s_d)^t}{T}. \tag{4}$$

Since $\sum_k U_j^T(s) = 1$, the dictatorial player's T-period expected utility in state s is

$$U_d^T(s) = 1 - \sum_{i \neq d} U_i^T(s) = 1 - \frac{1}{T} \sum_{t=1}^{T} (1 - s_d)^t. \tag{5}$$

Letting $T \to \infty$ yields

$$\lim_{T\to\infty} U_d^T(s) = 1, \quad \text{and} \quad \lim_{T\to\infty} U_i^T(s) = 0, \tag{6}$$

for any $i \neq d$. Note that the dictator's expected utility in the limit converges to 1 even when his status quo share is only slightly larger than $1/2$. The dictator becomes a tyrant as soon as he becomes a proposer and receives 1 for all subsequent periods. Similarly, a non-dictator's expected utility is zero in the limit even if his status quo share is only slightly smaller than $1/2$.

The next lemma characterizes the players' unique equilibrium behavior in collegial states. Recall that in collegial states, the player with a share of $1/2$ is called a *collegium player* and all other players are called *noncollegium players*.

Lemma 2 (Collegial States)**.** *In any equilibrium, proposal strategies in collegial states are as follows: The collegium player proposes the status quo, and all noncollegium players propose an oligarchic division with the collegium player and the proposer. All proposals are accepted by a decisive coalition.*

Lemma 2 implies that if the current state is collegial, then the environment always becomes oligarchic in the long run. The crucial intuition is that the collegium player is never able to pass a proposal that would make him a dictator. This is because the other players expect that any dictatorial states would eventually lead to the tyrannical state, leading to zero long-run payoffs for any nondictatorial player.

To obtain a brief intuition, let us analyze a noncollegium player's incentive to accept a dictatorial proposal. According to the strategies described in Lemma 2, the T-period expected utility of a noncollegium player i is recursively written as

$$U_i^T(s) = \frac{u_i(s) + (T-1)v_i^{T-1}(s)}{T} = \frac{1}{T}\left(s_i + (T-1)\frac{U_i^{T-1}(s) + s_i}{2}\right).$$

With the initial condition $U_i^1(s) = s_i$, it follows that $U_i^T(s) = s_i$ for any T. Then, combining this with (6), we have

$$\lim_{T\to\infty} T(U_i^T(s) - U_i^T(y)) > 0 \text{ for all } s \in X_{C_{-i}} \text{ and } y \in X_{D_{-i}}.$$

In other words, noncollegium players strictly prefer the status quo s to all other player's dictatorial divisions. In equilibrium, noncollegium players envision the possibility, however small it is, of becoming an oligarch and receiving $1/2$ for an infinite number of periods. In contrast, a dictatorial state eventually becomes tyrannical with probability one. Thus, any possible short-term gain from a dictatorial division is outweighed by its long-term cost. Given that any dictatorial division is rejected, the status quo provides the collegium player the maximum payoff among all socially acceptable divisions. Then the status quo bias assumption implies that he proposes to maintain the status quo wealth and power distribution. As a result, players never invite a dictator or tyrant in equilibrium, and the long-run outcome of a collegial state is a permanent oligarchy of two players.

It remains to characterize the equilibrium behavior in noncollegial states. The next lemma describes the long-run equilibrium outcome when the initial state is noncollegial.

Lemma 3 (Noncollegial States)**.** *In any equilibrium, if the initial state is noncollegial, the long-run outcome is a permanent oligarchy of two players.*

Lemmas 1 (item (iii)), 2 and 3 imply that unless the majority of the initial bargaining power is concentrated in a single player, bargaining with endogenous power never reaches a dictatorship and always ends in an oligarchy of two players.

The first part of the proof of Lemma 3 shows that there cannot be a direct transition from X_{NC} to either X_D or X_T. The key intuition is that in a noncollegial state, a player can exchange his bargaining position with any other player through his own proposal. To see this, consider a state $s \in X_{NC}$ and two players $i, j \in I$ with $s_i, s_j > 0$. Recall that the status quo s is unanimously accepted. Now, consider a proposal z such that $z_j = s_i$, $z_i = s_j$, and $z_k = s_k$ for all $k \neq i, j$. By symmetry, z is accepted by all players $k \neq i, j$. Then, since at least one player i or j prefers z to the status quo, z is socially accepted. Thus, player j can always propose z and successfully exchange his bargaining position with player i.

Now, suppose to the contrary that there exists player $i \in I$ who successfully passes his dictatorial proposal. Then, all other players also envision such a possibility of becoming a dictator by exchanging bargaining position with player i. Therefore, they strictly prefer the status quo to the dictatorial proposal made by player i, leading to a contradiction.

The first part implies that the best proposal a player can make is either oligarchic or collegial, in which the proposer takes 1/2, and that any noncollegial outcome is strictly suboptimal. In the second part of the proof, we show that there exists at least one player who is able to successfully pass either an oligarchic or a collegial offer by consolidating the shares among a certain decisive coalition. Therefore, any noncollegial state eventually becomes oligarchic, and the bargaining power becomes concentrated in two players.

We are now prepared to present the main result of the paper.

Theorem 1. *There exists an equilibrium of the game. In any equilibrium, the following holds:*

(i) If the initial state is dictatorial or tyrannical, the long-run outcome is a permanent tyranny of a single player. The dictator or tyrant in the initial state becomes a permanent tyrant.

(ii) If the initial state is not dictatorial or tyrannical, the long-run outcome is a permanent oligarchy of two players who equally share all wealth and power. If the initial state is oligarchic, the oligarchy perpetuates. If the initial state is collegial, the collegium player and the first noncollegium proposer constitute the permanent oligarchy.

Given Lemmas 1-3, it suffices to demonstrate the existence of the equilibrium to prove Theorem 1. Moreover, Lemmas 1-3 characterize the unique equilibrium profile in any $s \in X \backslash X_{NC}$. In Appendix B, we show its existence by constructing a candidate profile in X_{NC} and verifying its optimality.

Theorem 1 implies that if the players do not discount the future, the outcome of repeated bargaining with endogenous status quo *and* bargaining power eventually stabilizes into either tyranny or oligarchy. The long-run outcome depends on the initial distribution of power. If there is a single player who is overwhelmingly stronger than the other players at the beginning ($x^0 \in X_D \cup X_T$), the powerful player eventually becomes a tyrant. In contrast, if the players are initially at a relatively equal position to each other ($x^0 \in X_O \cup X_C \cup X_{NC}$), there is no tyranny in the long run: Instead, an oligarchy with two players emerges and perpetuates. Theorem 1 also identifies who constitutes the permanent oligarchy if the initial state is oligarchic or collegial.

For noncollegial states, Theorem 1 does not specify which players become permanent oligarchs. It is straightforward that if there are three players with positive shares, the first proposer always induces the permanent oligarchy with himself and one of the other players. However, the following example demonstrates that if there are more than three players, the first proposer may fail to become an oligarch.

Example 1. Consider a noncollegial state $s = (\frac{4}{13}, \frac{4}{13}, \frac{4}{13}, \frac{1}{13})$. Consider a proposal strategy $\hat{\mu}(s)$ such that for $i, j = 1, 2, 3$,

$$\hat{\mu}_i(y^j|s) = \begin{cases} \frac{1}{2} & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases}$$

where $y^1 = (\frac{1}{2}, \frac{1}{2}, 0, 0), y^2 = (0, \frac{1}{2}, \frac{1}{2}, 0), y^3 = (\frac{1}{2}, 0, \frac{1}{2}, 0)$. Note that we have not specified $\hat{\mu}_4(s)$. Recall that by Lemma 3, the highest payoff a player can achieve in any noncollegial state is 1/2. Since y^i $(i = 1, 2, 3)$ gives each coalition member a payoff of 1/2, it is accepted by a decisive coalition, and thus it is the proposer's optimal proposal. Furthermore, under $\hat{\mu}(s)$, the next period's state is oligarchic, which is an absorbing state. Therefore, there must exist an equilibrium in which the proposal strategy for players 1-3 are given as $\hat{\mu}(s)$.

Note that under $\hat{\mu}(s)$, any player $i = 1, 2, 3$ becomes an oligarch with probability of at least 8/13. This provides a lower bound on player i's T-period expected utility at state s:

$$U_i^T(s) \geq \frac{1}{T}\left(\frac{4}{13} + (T-1)\frac{4}{13}\right) = \frac{4}{13}.$$

Now we claim that at state s, player 4 cannot guarantee himself a place in the permanent oligarchy, even if he is selected as a proposer. To see this, note that player 4 needs at least two other players to form a decisive coalition, and thus player 4 offers at least $2U_i^T(s) \geq 8/13$ to the other players. However, the remainder is strictly less than half, which implies that player 4 cannot be an oligarch nor a collegium player in the next period by his proposal.■

In the constructed equilibrium in Appendix B, it takes at most two state transitions for any noncollegial state to become either oligarchic or collegial.[12] Thus, either the first or the second proposer secures a place in the permanent oligarchy. It remains to be seen whether the same property holds for any equilibrium of the game.

Considering the connection between the canonical Baron-Ferejohn model and the model analyzed in this article, the uniqueness of equilibrium and expected utility is of particular interest. As the following example demonstrates, however, there generally exist multiple equilibria associated

[12]Note that in our equilibrium, a noncollegium state may experience multiple state transitions before it becomes dynamically stable (oligarchic). This property contrasts with Acemoglu et al. (2012), who predict that any state becomes dynamically stable by a single state transition. This difference is due to stochastic proposer selection assumed in our model, in contrast to the deterministic proposer in Acemoglu et al. (2012). For example, consider a state $s' = (1/4, 1/4, 1/4, 1/4)$. Note that a direct transition from s' to an oligarchic state is not possible. Moreover, if the proposer sequence is deterministic in any state, then s' would be dynamically stable as there always exist some voters who lose the chance to propose in the new state. However, stochastic proposer selection can distribute the benefits of state transition to multiple voters, enabling a state transition from s' to a noncollegium state with three players.

with different expected utilities.

Example 2. Recall that in any equilibrium, if the current state is noncollegial with three positive-share players, then the next period's state must be in X_O in which the proposer becomes an oligarch. Moreover, since any $s \in X_O$ is an absorbing state, *any* oligarchic division by the proposer could constitute an equilibrium, potentially implying equilibrium multiplicity.

For example, consider $I = \{1, 2, 3\}$ and a noncollegial state $s = (\frac{4}{9}, \frac{3}{9}, \frac{2}{9})$. Let $\mu(s)$ and $\hat{\mu}(s)$ be two proposal strategies defined in s such that

$$\mu_i(s) = y^i,$$
$$\hat{\mu}_1(s) = \hat{\mu}_2(s) = y^1, \hat{\mu}_3(s) = y^3,$$

where $y^1 = (\frac{1}{2}, \frac{1}{2}, 0), y^2 = (0, \frac{1}{2}, \frac{1}{2}), y^3 = (\frac{1}{2}, 0, \frac{1}{2})$. Let σ and $\hat{\sigma}$ be the proposal strategies where σ assigns $\mu(s)$ and $\hat{\sigma}$ assigns $\hat{\mu}(s)$. Then since any of y^i leads to permanent oligarchy, both σ and $\hat{\sigma}$ are the equilibrium proposal strategies. Furthermore, player 1's expected utilities in state s under σ and $\hat{\sigma}$, respectively, yields

$$U_1^{T,\sigma}(s) = \frac{1}{T}\left(\frac{4}{9} + (T-1)\frac{1}{2}\left(\frac{4}{9} + \frac{2}{9}\right)\right) = \frac{1}{3} + \frac{1}{9T}$$
$$U_1^{T,\hat{\sigma}}(s) = \frac{1}{T}\left(\frac{4}{9} + (T-1)\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{18T},$$

implying that $U_1^{T,\sigma}(s) \neq U_1^{T,\hat{\sigma}}(s)$ when $T \to \infty$.■

The multiplicity of equilibria stems from the endogenous limit on socially acceptable proposals in noncollegial states. Specifically, a proposer in noncollegial states cannot take more than half of the total share because the other players never approve dictatorial proposals. However, there may exist multiple proposals (and infinite mixtures of those) in which the proposer's share is 1/2. In Example 2, player 1 can choose either player 2 or 3 as a coalition partner with any probability in his optimal proposal. Nevertheless, it should be emphasized that if a bargaining environment is initially noncollegial, the subsequent bargaining process always consolidates the wealth and power into a permanent oligarchy of two players.

4 Discussion

In this section, we discuss the role of the two main assumptions of our model.

4.1 Discounted Payoffs

One of the most significant assumptions of our main model is the no discounting of the future payoffs. To see the role of discounting, consider the other extreme case in which the players are completely myopic. Then, each player's expected utility is simply his status quo share, $U_i(s) = s_i$, and thus player i approves any proposal y with $y_i \geq s_i$. This myopic behavior invites a dictator even when the initial state is not dictatorial. For instance, if the initial state is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, the first proposer immediately becomes a dictator. An interesting question concerns the case between the two extreme ones: Can players with discounted payoffs prevent the transition to dictatorship when the initial power distribution is relatively equal?

To answer this question, let us first define the game with discounted payoffs. The bargaining procedure is analogous to the main model in Section 2. A group $I = \{1, \dots, n\}$ of players divides a dollar in each period $t = 1, \dots, \infty$. A state $s \in \bar{X} \subseteq X$ in each period represents the reversion point, recognition probability and voting weights (later, we formally define the set of states $\bar{X}$). At the beginning of each period, a randomly selected player makes a proposal $x \in \bar{X}$, and then each player votes for or against the proposal. The proposal x is passed and becomes the next period's state if it obtains more than half of the total voting weights; the status quo s remains in effect otherwise. In contrast to the main model, the players discount the future by a common factor $\delta < 1$.

Let $\sigma = \{(\mu_i, A_i)\}_{i \in I}$ be a Markov strategy profile defined in Section 2. Then, player i's expected utility $U_i^\sigma(s)$ in state s is recursively defined as

$$U_i^\sigma(s) = (1-\delta)s_i + \delta \sum_{j=1}^{n} s_j \sum_y \mu_j(y|s) \left(U_i^\sigma(y) \mathbb{1}_{A(s)}(y) + U_i^\sigma(s) \mathbb{1}_{X \setminus A(s)}(y) \right),$$

where $A(s)$ is the social acceptance set in state s and $\mathbb{1}$ is the indicator function. The equilibrium notion is the same as defined in Definition 1, with $x \succeq_{i,\sigma} y$ if and only if $U_i^\sigma(x) \geq U_i^\sigma(y)$.

Recall that in the main model, noncollegium players never accept a dictatorial offer in equilibrium. In the model with discounting, however, Proposition 1 below states that the equilibrium exhibits the same behavior *only if a noncollegium player's status quo share is not too small* relative to the discounting factor. If a player's current share is too small, the short-term gain from accepting a dictatorial offer may outweigh the long-term cost.

To see this, let us first calculate the players' payoffs in a dictatorial state $y \in X_{D_1}$ with $y_1 > 1/2$ and $y_i = 1 - y_1$ for some $i \neq 1$. Then, the expected payoff of the dictator (player 1) satisfies

$$U_1(y) = (1-\delta)y_1 + \delta(y_1 + (1-y_1)U_1(y)).$$

Using this, we calculate the payoffs of player 1 and i, which are

$$U_1(y) = \frac{y_1}{1-\delta(1-y_1)}, \quad U_i(y) = 1 - U_1(y) = \frac{(1-\delta)(1-y_1)}{1-\delta(1-y_1)}. \tag{7}$$

Now, let us verify the condition in which the profile in Lemma 2 holds as an equilibrium. Fix any $s \in X_{C_1}$. Then, given the profile in Lemma 2, the expected payoff of a noncollegium player $i \neq 1$ in state s is recursively written as

$$U_i(s) = (1-\delta)s_i + \delta\left(\frac{s_i}{2} + \frac{U_i(s)}{2}\right),$$

which simply leads to $U_i(s) = s_i$. Combining this with (7) implies that player i rejects player 1's dictatorial offer y, with $y_i = 1 - y_1$, if

$$s_1 \geq \frac{(1-\delta)(1-y_1)}{1-\delta(1-y_1)}.$$

Since $y_1 > 1/2$, player i rejects any dictatorial offer if

$$s_i > \frac{1-\delta}{2-\delta}. \tag{8}$$

The above condition shows the tradeoff associated with rejecting the dictatorial offer. The left-hand side of (8) captures player i's payoff from the status quo. Under the profile in Lemma 2, player i could become a permanent oligarch with a probability proportional to the current share. In contrast, the right-hand side of (8) is his maximum payoff from accepting a dictatorial offer. Note that the right-hand side converges to 0 as $\delta \to 1$: Even when player i receives a share just below 1/2 today, he will eventually lose the entire share to the dictator in the subsequent bargaining. For any $\delta < 1$, however, (8) is violated for sufficiently small s_i: A player with a small status quo share may find it difficult to resist a dictatorial offer, and thus a collegium state may become dictatorial.

With this intuition, let us provide a condition under which the results in Section 3 extend to the model with discounted payoffs. Let $I_s^+ = \{i \in I : s_i > 0\}$ be the set of players who have a nonzero share in state s. Then, for $\kappa \geq 0$, define $X_\kappa \subseteq X$ as

$$X_\kappa = \{s \in X : s_i > \kappa \text{ for all } i \in I_s^+\}.$$

Let $\Gamma_\kappa(\delta)$ be our bargaining game with endogenous power and discounted payoffs in which the set of states $\bar{X} = X_\kappa$ and the discounting factor is δ.

Proposition 1. *For any $\delta < 1$, then there exists $\bar{\kappa} > 0$ such that for any $\kappa > \bar{\kappa}$, any equilibrium in $\Gamma_\kappa(\delta)$ satisfies the following properties:*

1. *For any tyrannical, dictatorial, oligarchic, and collegial states, the equilibrium behavior is uniquely determined as one described in Lemmas 1 and 2.*

2. *For any noncollegial states, the equilibrium long-run outcome is a permanent oligarchy.*

Moreover, $\bar{\kappa} \to 0$ as δ goes to one.

If the status quo share of any player—if they have a nonzero share—is higher than a certain threshold, then they have a sufficiently large long-term incentive to reject any dictatorial offer. This 'minimum-share' assumption can be explained as a threshold on the player's wealth to have any decision-making power in the bargaining process.

Our minimum-share restriction on X is closely related to the 'finite policy space' assumption of Diermeier and Fong (2011). They consider a repeated bargaining game with a persistent agenda-setter and show that the agenda-setter's power is limited under the endogenous status quo. In their result, the discrete policy space assumption is crucial, as it prevents voters from choosing a policy with a slightly worse long-run outcome when it provides a large short-run benefit. In our model, a similar issue arises for a sufficiently small s_i.[13]

If we do not impose the minimum-share restriction on X, then the equilibrium outcome with discounted payoffs would be more likely to feature a long-run tyranny than the model with no discounting. For example, if the initial state is in $X_C \cap X_\kappa^C$, the collegium player eventually becomes a tyranny with probability one. However, when the initial state is in X_κ, other players would have

[13] Indeed, the results in Proposition 1 can also be obtained in a model with a properly discretized state space. For a detailed analysis, please see the previous version of this paper (Jeon, 2015).

stronger incentives to reject any offer $x \in X_\kappa^C$, knowing that accepting the offer may lead to a long-run tyranny. This intuition implies that Theorem 1 may be robust under discounted payoff as long as the initial state is in X_κ. It remains to be seen whether this intuition holds for any equilibrium.

4.2 General Voting Thresholds

Thus far, we have maintained the majority voting rule assumption, raising questions regarding its relationship with the equilibrium outcome of a perpetual oligarchy. This section analyzes the equilibrium under general supermajority voting thresholds in a three-player bargaining game. We show that as the voting thresholds increase, the long-run equilibrium outcome is more likely to be equitable, and the non-tyranny long-run outcomes become more robust to a perturbation in the environment.

Consider the main model in Section 2 with three players ($|I| = 3$) who do not discount future. Assume that every proposal needs strictly more than $q \in [1/2, 1)$ of the total voting weights. Note that the partitions of states given in Definition 2 naturally extend to the model with a general voting threshold. Let $X_T^q, X_D^q, X_O^q, X_C^q$ and X_{NC}^q be the corresponding set of states in a model with a voting threshold q. Importantly, the set of oligarchic states X_O^q may consist of two types: (a) one in which two players form an oligarchy and (b) one in which each of the three players has veto power.

Define X_2^q and X_3^q ($X_3^q \subset X_2^q \subset X$) as

$$X_2^q = \{x \in X | x_i \leq q \text{ for all } i\}, \text{ and } X_3^q = \{x \in X | x_i \geq 1 - q \text{ for all } i\}.$$

It is easy to check that $X_2^q = X \backslash (X_T^q \cup X_D^q)$ and that X_3^q is a strict subset of X_O^q in which each of three players has veto power.

The next proposition identifies the equilibrium long-run outcome depending on the initial state.

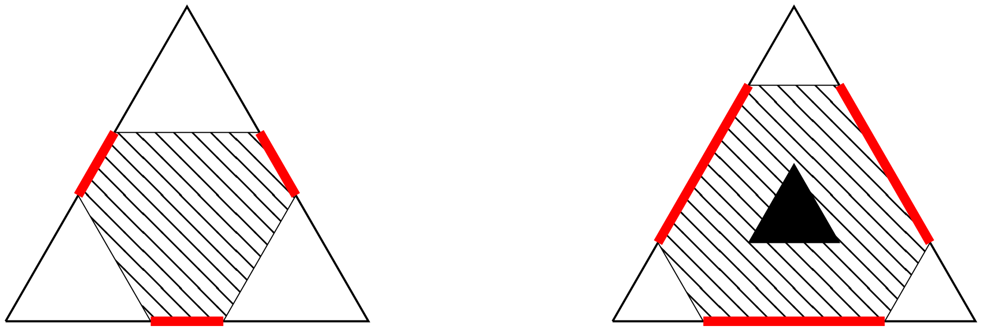
Proposition 2. *In any equilibrium with a voting threshold q, the following holds:*

(i) For any $x_0 \in X \backslash X_2^q$, the long-run outcome is tyranny.

(ii) For any $x_0 \in X_2^q \backslash X_3^q$, the long-run outcome is an oligarchy in which two players divide the entire share.

(iii) For any $x_0 \in X_3^q$, the initial state becomes the long-run outcome, an oligarchy in which each of three players has veto power.

Figure 2 illustrates how X_2^q and X_3^q change depending on the value of q. First, X_2^q expands as



(a) $q = 0.6$ (b) $q = 0.75$

Figure 2: A graphical illustration of Proposition 2 for different values of q. If $x_0 \in X \backslash X_2^q$ (white region), then the long-run outcome is a tyranny (a vertex). For any $x_0 \in X_2^q \backslash X_3^q$ (striped region), the state eventually becomes an oligarchic state in which two players share all the wealth (thick red lines on the edges of X). Any initial state in X_3^q (dark region) becomes a long-run outcome.

q increases, and thus, the set of initial states that become a tyranny shrinks. Intuitively, under a higher voting threshold, it becomes more difficult for any player to solely control the decision-making process. Second, the set X_3^q appears when q becomes greater than 2/3. For any $s \in X_3^q$, a proposal is passed only under unanimous support. Thus, any new proposal is rejected because it would harm at least one player, and the status quo remains as the long-run outcome. Note that almost every status quo becomes the long-run outcome (i.e., $X \backslash X_3^q$ shrinks) as q converges to one.

Our analysis in this section adds to the discussion regarding the robustness of the long-run outcome. In the main model, a small perturbation from oligarchy could result in a party's share strictly greater than 1/2, eventually leading to tyranny. However, the long-run oligarchy outcomes in Proposition 2 (thick red lines on the edges of X in Figure 2) generically do not have such property, making them more robust to perturbation.[14]

5 Concluding Remarks

The results of this paper suggest the prevalence of social inequality when it translates to ability to affect rules on resource distribution in the future. In our model, players who do not discount future successfully resist dictatorship. However, the alternative is a perpetuating oligarchy, which still exhibits a severe concentration of power. Most players lose the opportunity to advance their interests because they have no proposal power; having zero voting weights means adverse outcomes

[14]More precisely, for any oligarchic state in which no oligarch has a share exactly equal to q, there is a sufficiently small $\epsilon > 0$ such that after any state perturbation of size less than ϵ, each oligarch still has veto power.

can never be changed. In other words, all but two players end up being politically irrelevant.

Democratization can be perceived as an unexpected shock to a society that disturbs and redistributes existing economic and political power configurations. Our model predicts that even after a field-leveling institutional shock, society eventually transitions back into a state with concentrated power. This result is consonant with the earlier cautions that democratic projects may serve to legitimately protect the privileged or end up merely replacing one elite group with another:

> Note that partners to such pacts extract private benefits from democracy and that they protect their rents by excluding outsiders from competition. Democracy turns out to be a private project of leaders of some political parties and corporatist associations, and oligopoly in which leaders of some organizations collude to prevent outsiders from entering. (Przeworski, 1992, p.124)

Our results in Section 4.2 suggest that societies that commit to a supermajority voting rule could maintain redistributed political power. However, whether this intuition is robust when the voting rule itself is endogenously determined remains to be seen.

We conclude by suggesting potential future research directions. First, it would be interesting to analyze the role of two innovations of the paper—endogenous voting weights and recognition probabilities—separately. For example, it is easy to see that Lemmas 1 and 2 hold in a model with endogenize voting weights but fixed recognition probabilities. In the equilibrium, the players with nonzero voting weights would always reject offers from zero-share players. However, it is not clear if we could obtain the same equilibrium characterization result for noncollegial states. Also, we conjecture that only the endogenous recognition probability may lead to an unequal outcome in the long run, given that the zero-share players accept a proposal when they are indifferent.

Second, one might consider a model in which the inequality of wealth and power negatively affects economic productivity (the amount of total available resources). In this case, players need to consider not only the long-term stability of their political power but also future productivity change. An oligarchy would still enable the oligarchs to retain their power but may not maximize their long-term payoffs. Third, existing bargaining models show that risk aversion is one of the primary sources that induce players to compromise. In our model, investigating the effect of risk aversion on the possibilities of stable coalitions larger than oligarchies represents be an interesting future research agenda.

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Appendix

The Appendix consists of two parts. Appendix A presents all omitted proofs except that of Theorem 1. Appendix B proves Theorem 1 by constructing an equilibrium profile for $s \in X_{NC}$.

A Omitted Proofs

Proof of Lemma 1. **Part (i):** Shown in the main text.

Part (ii): Assume without loss of generality that player 1 is the dictator. Note that given part (i), player 1 strictly prefers his tyrannical share over any other bargaining outcome. Moreover, for any state $s \in X_{D_1}$, the social acceptance set coincides with player 1's acceptance set, i.e., $A(s) = A_1(s)$. Therefore, in any equilibrium, in any state $s \in X_{D_1}$, player 1 offers his tyrannical share and accepts the proposal himself.

Given this, we claim that in any equilibrium σ^*, for any $i \in I$, $y \in X$ and $s \in X_{D_1}$,

$$\liminf_{T\to\infty} T(U_i^{\sigma^*,T}(y) - U_i^{\sigma^*,T}(s)) = \limsup_{T\to\infty} T(U_i^{\sigma^*,T}(y) - U_i^{\sigma^*,T}(s)), \tag{9}$$

whenever the limits exist. Observe that in any equilibrium, from any $s \in X_{D_1}$, the equilibrium outcome has an absorbing state in which player 1 is tyranny. This is because player 1 would reject any offer that would result in a non-tyrannical long-run state, which is player 1's most preferred outcome. Therefore, the limits in (9) exist if and only if the state y induces player 1's tyrannical long-run outcome, and in this case the inferior and superior limits coincide.

It remains to characterize the equilibrium offers of non-dictatorial players. We partition X_{D_1} into two disjoint sets $X^1_{D_1}$ and $X^2_{D_1}$, where $X^1_{D_1}$ ($X^2_{D_1}$) is the set of states in which one player (two or more players) other than player 1 has a nonzero share.

First, consider a state in $X^1_{D_1}$. Without loss of generality, assume that player 2 has a positive share, and fix a state $s' = (s'_1, 1 - s'_1, 0, \dots, 0)$ with $s'_1 > 1/2$. We need to show that player 2 offers the status quo at state s' in any equilibrium. Suppose to the contrary that there exists an equilibrium σ^* in which player 2 offers some $y \neq s'$ at state s', and that player 1 accepts y. Since it must be that y also induces the player 1's tyrannical long-run outcome, (9) holds. Our status-quo bias assumption implies that player 2 must strictly prefer y to s', that is,

$$\lim_{T\to\infty} T(U_2^{\sigma^*,T}(y) - U_2^{\sigma^*,T}(s')) > 0, \tag{10}$$

by overtaking criterion (by (9), we replace the lim inf with lim).

Recall from (5) that if player 2 always offers the status quo s', then he obtains the payoff of $\underline{U}_2^T(s') \equiv \frac{1}{T}\sum_{t=1}^T (a - s_1')^t$. Similarly, player 1 can guarantee himself a payoff of $\underline{U}_1^T(s') \equiv 1 - \underline{U}_2^T(s')$ by always rejecting any offer $x \neq s'$. Then it must be that $\lim_{T\to\infty} T(U_1^{\sigma^*,T}(s') - \underline{U}_1^T(s')) \geq 0$ and $\lim_{T\to\infty} T(U_2^{\sigma^*,T}(s') - \underline{U}_2^T(s')) \geq 0$. Combining with (10), we have

$$\begin{aligned}
\lim_{T\to\infty} T(U_1^{\sigma^*,T}(s') - U_1^{\sigma^*,T}(y)) &\geq \lim_{T\to\infty} T(\underline{U}_1^T(s') - U_1^{\sigma^*,T}(y)) \\
&\geq \lim_{T\to\infty} T(U_2^{\sigma^*,T}(y) - \underline{U}_2^T(s')) \\
&\geq \lim_{T\to\infty} T(U_2^{\sigma^*,T}(y) - U_2^{\sigma^*,T}(s')) > 0,
\end{aligned}$$

where the second inequality is from the fact that $U_1^{\sigma^*,T}(y) + U_2^{\sigma^*,T}(y) \leq 1$ and $\underline{U}_1^T(s') + \underline{U}_2^T(s') = 1$ for all $y \in X$ and T. However, the above inequality implies that player 1 would reject y, a contradiction.

The above discussion implies that for any $s \in X_{D_1}^1$, player 1's T-period equilibrium payoff is given by (5): with a simplified notation (omitting σ^*), we have

$$U_1^T(s) = 1 - \frac{1}{T}\sum_{t=1}^{T}(1 - s_1)^t. \tag{11}$$

Note that $U_1^T(s)$ is continuous and monotonically increasing in s_1.

Next, consider a state in $X_{D_1}^2$. Fix a state $s'' \in X_{D_1}^2$ with $s_1'' > 1/2$. Define $\Xi(s_1)$ as a subset of $X_{D_1}^1$ in which player 1's share is s_1.

We claim that player 1 is indifferent between s'' and any $s \in \Xi(s_1'')$. It is clear that player 1 weakly prefers s'' to s, because he can guarantee the payoff of $U_1^T(s)$ (equation 11) in state s'' by always rejecting other players' offers. Now suppose to the contrary that player 1 strictly prefers s'' to s. Then it must be that in state s'', some player $i \neq 1$ makes an offer y such that player 1 strictly prefers y to s''. Let $s_1^\dagger \in (s_1'', 1]$ be such that for any $z \in \Xi(s_1^\dagger)$, we have $\lim_{T\to\infty} T(U_1^T(z) - U_1^T(y)) = 0$. Note that the claim associated with equation (9) enables us to use lim instead of lim inf, and that the continuity of (11) in s_1 guarantees the existence of $s_1^\dagger$.

By construction, player 1 weakly prefers z to y. Moreover, we can show that player i also weakly prefers z to y: Since $U_1^T(z) + U_i^T(z) = 1$ and $U_1^T(y) + U_i^T(y) \leq 1$ for all T,

$$\lim_{T\to\infty} T(U_i^T(z) - U_i^T(y)) \geq \lim_{T\to\infty} T(U_1^T(y) - U_1^T(z)) = 0,$$

where the last equality comes from the definition of $s_1^\dagger$ and z. However, for sufficiently small $\varepsilon > 0$, the continuity of (11) implies that player 1 strictly prefers any offer in $\Xi(s_1^\dagger - \varepsilon)$ to s''. Therefore, player i has a profitable deviation (from offering y) to offering $s_1^\dagger - \varepsilon$ to player 1 and remainder to himself, leading to a contradiction.

The above analysis implies that player 1 accepts any offer x with $x_1 \geq z_1$. Given this, any player $j \neq 1$ finds it strictly optimal to offer x with $x_1 = z_1$ and $x_j = 1 - z_1$, showing the desired result.

Part (iii): Consider a state $s \in X_O$ where players i and j form the oligarchy. Note that either player can guarantee himself a payoff of 1/2 by always offering the status quo and rejecting the other's offer. Now suppose to the contrary that there exists an equilibrium σ^* in which player j—without loss of generality—offers some $w \neq s$. Since player j must strictly prefer w to s, we have

$$\liminf_{T\to\infty} T\left(U_j^{\sigma^*,T}(w) - \frac{1}{2}\right) > 0. \tag{12}$$

However, since $U_i^{\sigma^*,T}(w) + U_j^{\sigma^*,T}(w) \leq 1$ for all T,

$$\begin{aligned}\liminf_{T\to\infty} T\left(U_i^{\sigma^*,T}(w) - \frac{1}{2}\right) &\leq \liminf_{T\to\infty} T\left(\frac{1}{2} - U_j^{\sigma^*,T}(w)\right) \\ &= -\limsup_{T\to\infty} T\left(U_j^{\sigma^*,T}(w) - \frac{1}{2}\right) < 0,\end{aligned}$$

where the last inequality is from (12). Therefore, player i has strict incentive to reject w, leading to a contradiction. ■

The following claim is useful in the proofs of Lemmas 2 and 3.

Claim 1. *In any equilibrium, there is no direct transition from a noncollegial state to either a dictatorial or tyrannical state.*

Proof of Claim 1. Suppose on the contrary that there exists an equilibrium and a state $s \in X_{NC}$ in which at least one player proposes and passes a dictatorial division y with the support of a decisive coalition. Without loss of generality, let player 1 be the proposer and player 2 belong to the decisive coalition that accepts y. Also, we assume that y gives a dictatorial share to player 1; slightly modifying the below proof shows the desired result if $y \in X_{D_i}$ for any i.

First, consider the case in which player 1 uses a pure strategy at s_1. Observe that player 1 is selected as a proposer with probability s_1, in which case he proposes y with probability one. By the definition of the continuation value (equation 3), this implies that

$$v_1^T(s) \geq s_1 U_1^T(y) \tag{13}$$

for all T. Next, as described in the main body, a permutation of s in which only s_1 and s_2 are switched must be in the social acceptance set. This provides a lower bound on player 2's continuation payoff: It must be that $v_2^T(s) \geq s_2 U_1^T(s)$ for all T.

From (4), player 2's payoff from accepting y is

$$U_2^T(y) = \frac{y_2 + y_2 \sum_{t=1}^{T-1}(1-y_1)^t}{T}.$$

However, his payoff from the status quo is

$$\begin{aligned}
U_2^T(s) &= \frac{s_2 + (T-1)v_2^{T-1}(s)}{T} \\
&\geq \frac{s_2 + (T-1)s_2 U_1^{T-1}(s)}{T} \\
&= \frac{s_2 + s_2(s_1 + (T-2)v_1^{T-2}(s))}{T} \\
&\geq \frac{s_2 + s_2(s_1 + (T-2)s_1 U_1^{T-2}(y))}{T} \\
&= \frac{s_2 + s_2\left(s_1 + (T-2)s_1\left(1 - \frac{1}{T-2}\sum_{t=1}^{T-2}(1-y_1)^t\right)\right)}{T},
\end{aligned}$$

where the last equality is from (5). Since $\lim_{T\to\infty} U_2^T(s) = s_1 s_2 > 0$ and $\lim_{T\to\infty} U_2^T(y) = 0$, player 2 strictly prefers s to y, a contradiction to the supposition that player 2 accepts y.

Now, suppose that player 1 uses a mixed strategy at s_1, i.e., player 1 proposes some $z \neq y$ with positive probability. Then (13) becomes $v_1^T(s) \geq \inf_{Support(\mu_1(s))} s_1 U_1^T(y)$. Since player 1 is indifferent between any $z \in \mu_1(s)$ and y, we have $\lim_{T\to\infty}(U_1^T(y) - U_1^T(z)) = 0$, and thus an argument similar to above shows that player 2 strictly prefers s to y, leading to a contradiction. ■

Proof of Lemma 2. Without loss of generality, assume that player 1 is the collegium player (i.e.,

$s_1 = 1/2$). The following claim is the main element of the proof:

Claim 2. *In any equilibrium, at any state in X_{C_1}, player 1 never proposes $x \in X_{D_1} \cup X_{T_1}$.*

Proof of Claim 2. It is straightforward that player 1's tyrannical share is never accepted. Now, suppose to the contrary that there exists an equilibrium and a collegial state $s \in X_{C_1}$ such that player 1 proposes $y \in X_{D_1}$ that is accepted by at least one player other than player 1. From (5), player 1's payoff from outcome y is

$$U_1^T(y) = 1 - \frac{1}{T}\sum_{t=1}^{T}(1 - y_1)^t.$$

Let $U_{-1}^T(y) = 1 - U_1^T(y)$ be the sum of payoffs of all players $i \neq 1$ under y. Then, $\lim_{T\to\infty} T U_{-1}^T(y) = (1 - y_1)/y_1$.

It is clear that player $i \neq 1$ never proposes player 1's tyrannical share. Also, Claim 1 implies that player 1 must reject any noncollegial offer. Finally, player 1 rejects any oligarchic offer, as accepting it would lead to permanent oligarchy (Lemma 1), while in state s, player 1 will be a tyrant with probability one. Therefore, it suffices to analyze the following two cases:

Case 1: *Some player $j \neq 1$ proposes a dictatorial share in state s.* Let z be player 1's most preferred share among the equilibrium offers made by $i \neq 1$. Then, player 1's continuation payoff is bounded by

$$v_1^T(s) \leq \frac{U_1^T(y)}{2} + \frac{U_1^T(z)}{2}.$$

Since z must be a dictatorial share, we have $\lim_{T\to\infty} T U_{-1}^T(z) = (1 - z_1)/z_1$.

We derive a contradiction by showing that player j has an incentive to deviate from offering z. To see this, consider player 1's incentive to accept z. Comparing $U_1^T(z)$ and $U_1^T(s)$, we have

$$\begin{aligned}
\lim_{T\to\infty} T(U_1^T(z) - U_1^T(s)) &\geq \lim_{T\to\infty} T U_1^T(z) - \left(\frac{1}{2} + (T-1)\left(\frac{U_1^{T-1}(y)}{2} + \frac{U_1^{T-1}(z)}{2}\right)\right) \\
&\geq \lim_{T\to\infty} T(1 - U_{-1}^T(z)) - \left(\frac{1}{2} + (T-1)\left(\frac{1 - U_{-1}^{T-1}(y)}{2} + \frac{1 - U_{-1}^{T-1}(z)}{2}\right)\right) \\
&= \lim_{T\to\infty} -T U_{-1}^T(z) + \frac{1}{2}\left(1 + (T-1)\left(U_{-1}^{T-1}(y) + U_{-1}^{T-1}(z)\right)\right) \\
&= \frac{1}{2}\left(1 + \frac{1 - y_1}{y_1} - \frac{1 - z_1}{z_1}\right),
\end{aligned}$$

which is strictly positive for any $y_1, z_1 > 1/2$. Therefore, player 1 strictly prefers to accept z.

However, then player j has a profitable deviation to slightly lower player 1's share and increase his share (recall from (11) that players' payoffs in dictatorial states are continuous), leading to a contradiction.

Case 2: *All player $i \neq 1$ proposes a collegial share in state s.* In this case, first we show that for any $x \in X_{C_1}$ and $y \in X_{D_1}$, player 1 strictly prefers y to x. Let $\bar{U}_1^T(X_{C_1}) = \sup_{x \in X_{C_1}} U_1^T(x)$. Note that our result in Case 1 implies that all player $i \neq 1$ proposes a collegial share at any $x \in X_{C_1}$. Therefore,

$$\bar{U}_1^T(X_{C_1}) \leq \frac{\frac{1}{2} + \frac{1}{2}(T-1)(\bar{U}_1^T(X_{C_1}) + U_1^{T-1}(z))}{T},$$

for some $z \in X_{D_1}$. Simplifying, using the fact that $U_{-1}^T(x) = 1 - U_1^T(x)$, and taking the limit yield

$$\lim_{T\to\infty} TU_{-1}^T(x) \geq 1 + \lim_{T\to\infty} TU_{-1}^T(z) > 1,$$

for any $x \in X_{C_1}$. However, $\lim_{T\to\infty} TU_{-1}^T(y) < 1$ for any $y \in X_{D_1}$, player 1 strictly prefers y to x.

Suppose that player k accepts player 1's dictatorial offer y at state s. Also, assume that player k offers $w \in X_{C_1}$. Then it must be that $\lim_{T\to\infty} TU_k^T(s) = \lim_{T\to\infty} TU_k^T(y) = (1-y_1)/y_1$. Moreover, the above argument implies that for a sufficiently small $\varepsilon > 0$, player 1 would accept an offer $z \in X_{D_1}$ in which $z_1 = 1/2 + \varepsilon$ and $z_k = 1 - z_1$. Recall that $\lim_{T\to\infty} TU_k^T(z) = (1-z_1)/z_1$. Since player k must find it optimal to make a collegial offer, it follows that $\lim_{T\to\infty} TU_k^T(w) \geq 1$.

It is clear that player 1 offers a dictatorial offer at state w; otherwise, player 1 would reject the transition to w. Moreover, player 1 never offers a nonzero share to player k at state w, since player k strictly prefer w to any dictatorial offer. Therefore,

$$U_k^T(w) \leq \frac{w_k + (T-1)(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot U_k^{T-1}(w))}{T}$$

Simplifying and taking the limit yield

$$\lim_{T\to\infty} TU_k^T(w) \leq 2w_k < 1.$$

However, this contradicts to the above argument that $\lim_{T\to\infty} TU_k^T(w) \geq 1$. ■

Claim 2 implies that player 1's expected payoff is no more than 1/2. Since his reservation value is 1/2, player 1 will always offer the status quo. For player $i \neq 1$, the optimal outcome is an oligarchy of player 1 and i, which player 1 will accept. ■

***Proof of Lemma 3*.** Claim 1 implies that the optimal outcome for any player in a noncollegial state is to be a collegial or oligarchic player. In the remainder of the proof, we show that for any noncollegial state, there exists a player whose collegial or oligarchic proposal is accepted by a decisive coalition. Therefore, for any noncollegial state, there is a positive probability that the next period's state becomes either collegial or oligarchic. Combining this with Lemma 2, we have our desired result.

Fix any $s \in X_{NC}$, and let $m \leq n$ players have positive status quo shares. Align the indices of players such that $s_1 \geq \ldots \geq s_m > 0$ and $s_i = 0$ for $i = m+1, \ldots, n$. If $s_i = s_j$ for all $i, j \leq m$, it is clear that every $i \leq m$ can take $1/2$ in a player's proposal by symmetry and the number of other players' votes necessary to pass a proposal.

Now suppose that $s_i \neq s_j$ for some $i, j \leq m$, which implies that $s_1 > s_m$. Let

$$h = \min\{k : \sum_{i \leq k} s_i \geq \frac{1}{2}\}$$

be the smallest index such that players 1 to h form a decisive coalition. If $\sum_{i \leq h} s_i > 1/2$, player 1 has at least two disjoint sets of coalition partners to form decisive coalitions: $C_1 = \{2, \ldots, h\}$ and $C_2 = \{h+1, \ldots, m\}$. In other words,

$$s_1 + \sum_{i \in C_1} s_i > \frac{1}{2}, \qquad \text{and} \qquad s_1 + \sum_{i \in C_2} s_i > \frac{1}{2}.$$

Since $\sum_{i \in I} U_i^T(s) = 1$, either $\sum_{i \in C_1} U_i^T(s) \leq 1/2$ or $\sum_{i \in C_2} U_i^T(s) \leq 1/2$ or both for any T. Thus, player 1 needs to pay less than or equal to $1/2$ of the total expected utility to the other players to pass his proposal (recall from Section 3 that $U_i^T(s) = s_i$ for any T and any oligarchic or collegial states, and thus player 1 can continuously distribute the utility). Therefore, there exists a proposal that makes player 1 either a collegium player or an oligarch in the next period, which player 1 finds strictly optimal.

If $\sum_{i \leq h} s_i = 1/2$, player 1 can again form decisive coalitions with the following two disjoint sets of players: $C_1 = \{2, \ldots, h, m\}$ and $C_2 = \{h+1, \ldots, m-1\}$ ($C_2 \cup \{i\}$ is a decisive coalition since $s_1 > s_m$), and the same argument above applies. Accordingly, any noncollegial state becomes a collegial or oligarchic state with a positive probability. By Lemma 2, such a process results in an oligarchy in the long run. ■

***Proof of Proposition 1*.** Let $X_{a,\kappa} = X_a \cap X_\kappa$ ($a = T, D, O, C, NC$) be the partitions of the state space in X_κ. Below, we show the desired results in each state partition.

Tyrannical, dictatorial, and oligarchic states: Demonstrating the statement for $X_{T,\kappa}, X_{D,\kappa}$ and $X_{O,\kappa}$ is straightforward. Indeed, a stronger result can be obtained for these states: for any $\delta < 1$ and $\kappa \geq 0$, the equilibrium behavior of $\Gamma_\kappa(\delta)$ in $X_{T,\kappa}, X_{D,\kappa}$ and $X_{O,\kappa}$ is identical to that described in Lemma 1.

Collegial states: To prove the statement for collegial states, we replicate our argument in the proof of Lemma 2. Assume without loss of generality that player 1 is the collegium player. First, we claim that a collegium player never offers a dictatorial share in any equilibrium. Suppose to the contrary that there exists an equilibrium in which for some state $s' \in X_{C_1,\kappa}$, player 1 makes $y \in X_{D_1,\kappa}$ that is accepted by a decisive coalition. Similar to the proof of Lemma 2, it is easy to show that player 1 would always reject other players' oligarchic offer at s'. Moreover, our proof for noncollegial states below (which does not need our results for collegial states) shows that player 1 must reject any noncollegial offer, as doing so would prevent him from becoming a dictator. Then, it suffices to consider the following two cases:

- ***Case 1:*** *there exists some player $j \neq 1$ who offers some $w \in X_{D_1,\kappa}$ in state s'.* First note that player 1 must be indifferent between w and s', i.e., $U_1(w) = U_1(s')$. If $U_1(w) > U_1(s')$, player j has an incentive to slightly lower the offer; if $U_1(w) < U_1(s')$, player 1 rejects w. Then it follows that any dictatorial offer made by player $i \neq 1$ must give w_1 to player 1.

  Now we claim that all player $i \neq 1$ must offer a dictatorial offer with w_1 as player 1's share. To see this, note that the above argument implies that player i has two options: either offer some $x \in X_{C_1,\kappa}$ or w. If player i offers x, then his payoff is strictly less than $1 - U_1(s')$. However, his payoff from offering w is $1 - U_1(w) = 1 - U_1(s')$, which is strictly better.

  Given this, player 1's indifference condition between accepting and rejecting w is

  $$U_1(w) = (1-\delta)\frac{1}{2} + \delta\left(\frac{1}{2}U_1(y) + \frac{1}{2}U_1(w)\right),$$

  From (7), $U_1(y) = y_1/(1-\delta(1-y_1))$. Therefore,

  $$U_1(w) = \frac{1}{2-\delta} - \delta(1-y) < \frac{1}{2-\delta}.$$

However, from $U_1(w) = w_1/(1-\delta(1-w_1))$ since $w \in X_{D_1,\kappa}$, which yields

$$w_1 = \frac{(1-\delta)U_1(w)}{1-\delta U_1(w)} < \frac{1}{2},$$

which contradicts the presumption that $w \in X_{D_1,\kappa}$.

- ***Case 2:*** *all $j \neq 1$ offer some $x \in X_{C_1,\kappa}$ in state s'.* First, we claim that for any $x \in X_{C_1,\kappa}$ and $z \in X_{D_1,\kappa}$, player 1 strictly prefers z to x. Let $\bar{U}_1(X_{C_1,\kappa}) = \sup_{x \in X_{C_1,\kappa}} U_1(x)$. Then

$$\bar{U}_1(X_{C_1,\kappa}) \leq (1-\delta)\frac{1}{2} + \delta\left(\frac{1}{2}\bar{U}_1(X_{C_1,\kappa}) + \frac{1}{2}\right) \Longrightarrow \bar{U}_1(X_{C_1,\kappa}) \leq \frac{1}{2-\delta}.$$

However, for any $z \in X_{D_1,\kappa}$, $U_1(z) = z_1/(1-\delta(1-z_1)) > 1/(2-\delta)$, and thus $z \succ_1 x$.

Now, suppose that player k accepts y at state s'. Also, assume that player k offers $w \in X_{C_1,\kappa}$. The above argument implies that player 1 would accept any offer in $X_{D_1,\kappa}$. Since player k must find it optimal to offer w, it follows that for any $z_1 > 1/2$,

$$U_k(w) \geq \frac{(1-\delta)(1-z_1)}{1-\delta(1-z_1)},$$

and thus $U_k(w) > (1-\delta)/(2-\delta)$.

It must be that player 1 offers a dictatorial offer at state w; otherwise, player 1 would reject w when it is offered by player k. Moreover, player 1 never offers a nonzero share to player k at state w, since player k strictly prefer w to any dictatorial offer. Finally, w provides the highest payoff to player k among any collegial offers that is accepted by player 1. Therefore,

$$U_k(w) \leq (1-\delta)w_1 + \delta\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot U_k(w)\right).$$

However, since $w_1 < 1/2$, it follows that $U_k(w) < (1-\delta)/(2-\delta)$, which leads to a contradiction.

Noncollegial states: First, we replicate the proof of Claim 1 to show that if $s \in X_{NC,\kappa}$, any offer in either $X_{T,\kappa}$ or $X_{D,\kappa}$ is never accepted in equilibrium. Suppose to the contrary that player 1 offers $y \in X_{D_1,\kappa}$ and player 2 accepts the offer. Then, the symmetry requirement implies that player 2 can exchange his bargaining position with player 1 by successfully proposing and offering s' with

$s'_1 = s_2$, $s'_2 = s_1$ and $s'_k = s_k$ for $k \neq 1, 2$. Therefore, player 2's payoff $U_2(s)$ bounded below by

$$\begin{aligned} U_2(s) &\geq (1-\delta)s_2 + \delta s_2 U_1(s) \\ &\geq (1-\delta)s_2 + \delta s_2((1-\delta)s_1 + \delta s_1 U_1(y)), \end{aligned}$$

where the second inequality comes from the fact that player 1 can propose and pass y if he is selected as a proposer. From (7), it follows that

$$U_2(s) - U_2(y) \geq Z \equiv (1-\delta)s_2 + \delta s_2\left((1-\delta)s_1 + \frac{\delta s_1 y_1}{1-\delta(1-y_1)}\right) - \frac{(1-\delta)(1-y_1)}{1-\delta(1-y_1)}.$$

Therefore, if $Z > 0$, then player 2 strictly prefers to reject y, reaching a contradiction. Note that Z is increasing in s_1, s_2 and y_1. Therefore, $Z > 0$ for any $s_1, s_2 > \kappa$ and $y_1 > 1/2$ if

$$\kappa((2-\delta)(1-\delta)(1+\delta\kappa) + \delta^2\kappa) - (1-\delta) > 0.$$

Solving this equation shows that this inequality is satisfied for all $\kappa > \hat{\kappa}$, where

$$\hat{\kappa} = \frac{-(2-\delta)(1-\delta) + \sqrt{(2-\delta)^2(1-\delta)^2 + 4\delta(2-2\delta+\delta^2)(1-\delta)}}{2\delta(2-2\delta+\delta^2)}$$

It is straightforward that $\hat{\kappa} \to 0$ as $\delta \to 1$.

For any $\delta < 1$, define $\bar{\kappa} = \max\{(1-\delta)/(2-\delta), \hat{\kappa}\}$. Then, the combination of (8) and the above argument implies that for any $\kappa > \bar{\kappa}$, in any equilibrium of $\Gamma_\kappa(\delta)$, if the initial state x_0 is in $X_{C,\kappa} \cup X_{NC,\kappa}$, the long-run outcome is never tyrannical or dictatorial.

Finally, the exact replication of the second part of the proof of Lemma 3 shows that the equilibrium long-run outcome of any noncollegial state is a permanent oligarchy. ■

Proof of Proposition 2 . **Part (i):** It is straightforward that for $s \in X \backslash X_2^q = X_T^q \cup X_D^q$, the equilibrium behavior is identical to one in Lemma 1.

Part (ii): First, a replication of the proof of Lemma 2 shows that the behavior in X_C^q is identical to that in the benchmark model. Next, consider the case with $x \in X_2^q \backslash (X_C^q \cup X_3^q)$. Again, a replication of the first part of Lemma 3 implies that there cannot be a direct transition from x to X_T^q or X_D^q. We claim that for any $s \in X_2^q \backslash (X_C^q \cup X_3^q)$, the next period's state is in X_O^q with two players. Assume without loss of generality that player 1 becomes the proposer and that player 2's payoff from the status quo is lower than that of player 3. Then, since his proposal is accepted by only one of the other players, his best strategy is to form a coalition of him and player 2 and offer player 2 a payoff exactly equal to his payoff from the status quo and take the rest (if it is strictly greater than q, then player 1 proposes $(q, 1-q, 0)$).

Part (iii): Suppose that $s \in X_3^q$. Then, since every player has veto power, no proposal is accepted unless it provides each player with a payoff exactly equal to the payoff from the status quo. Then, since the proposer is indifferent between any proposals in the social acceptance set, by the status quo bias requirement, he offers the status quo division. ■

B Proof of Theorem 1: Equilibrium Existence

In this part, we prove the existence of an equilibrium of our main model by the constructive method. From Lemmas 1 and 2, equilibrium strategies in tyrannical, dictatorial, oligarchic and collegial states are uniquely determined. Therefore, it remains to construct a strategy profile in noncollegial states and demonstrate its optimality.

Recall that $I^s_+ \subseteq I$ is the set of players with nonzero shares in state s. Partition X_{NC} into two disjoint sets X^e_{NC} and X^{ne}_{NC}, where X^e_{NC} is the set of noncollegial states in which every player having a positive share can form a decisive coalition composed of two players. Formally,

$$X^e_{NC} = \{s \in X_{NC} | \text{ for any } i \in I^s_+, \text{there exists } j \neq i \text{ such that } s_i + s_j > 1/2\},$$

and $X^{ne}_{NC} = X_{NC} \backslash X^e_{NC}$. In the remainder of this section, we divide our analysis into X^e_{NC} (Section B.1) and X^{ne}_{NC} (Section B.2) and construct an equilibrium profile in each set.

We introduce new notations that we utilize for our proof. Recall that $\mathcal{L}^s$ is the set of all decisive coalitions in state s. Let $\mathcal{L}^s_i$ be the set of decisive coalitions that include player $i \in I^s_+$. Similarly, let $\underline{\mathcal{L}}^s$ and $\underline{\mathcal{L}}^s_i$ be the set of all minimal decisive coalitions in state s and the set of its elements that include player i, respectively.

B.1 Equilibrium Profile for $\mathbf{s \in X^e_{NC}}$

We begin our equilibrium construction with X^e_{NC}. The following claim identifies the set of all minimal decisive coalitions under state $s \in X^e_{NC}$.

Claim 3. *For $s \in X^e_{NC}$. Then, there exists $i \in I^s_+$ such that the set of all minimal decisive coalitions $\underline{\mathcal{L}}^s$ under state s consists of the following:*

- $L_{ij} \equiv \{i, j\}$ *for all* $j \in I^s_+ \backslash \{i\}$, *and*
- $J \equiv I^s_+ \backslash \{i\}$.

Proof of Claim 3. The case with $|I^s_+| = 3$ is straightforward since the coalitions $\{1, 2\}, \{1, 3\}, \{2, 3\}$ are all minimally decisive. Now, consider the case in which $|I^s_+| \geq 4$. Without loss of generality, suppose that $s_1 + s_2 > 1/2$. Then, $s_j + s_k < 1/2$ for any two players $j, k \in I^s_+ \backslash \{1, 2\}$. Thus, there must exist $i \in \{1, 2\}$ such that $s_i + s_j > 1/2$. Again, without loss of generality, assume that $i = 1$. Then, $s_2 + s_k < 1/2$ for all $k \neq I^s_+ \backslash \{1, j\}$, which implies that $L_{1k} = \{1, k\} \in \underline{\mathcal{L}}^s$ for all $k \in I^s_+ \backslash \{1\}$.

Let player i be the player who can form a decisive coalition of size two with every other player. Then, $J \equiv I_+^s \backslash \{i\}$ is the unique minimal decisive coalition that does not include player i. Suppose to the contrary that $J \backslash \{k\}$ is decisive for some $k \in J$. However, it must then be that $s_1 + s_k < 1/2$, which contradicts the assumption that $L_{ij} \in \underline{\mathcal{L}}^s$ for all $j \in I_+^s \backslash \{i\}$. ■

Given Claim 3, we construct a candidate profile as follows. We divide a player's proposal strategy into two parts: a player's *coalition choice strategy* that describes the probability of choosing a certain coalition and a *proposal rule* that determines the share distribution within a chosen coalition.

For the remainder of this subsection, we designate i as the player who can form a decisive coalition with every other player. For each $k \in I_+^s$, construct a coalition formation strategy $\xi_k^s \in \Delta \underline{\mathcal{L}}_k^s$ as follows: For all $j \in J$,

$$\xi_i^s(L_{ij}) = \frac{s_j}{s_i} \left(1 - \frac{1 - 2s_i}{1 - s_i} \sum_{h \in J \backslash \{j\}} \frac{s_h}{1 - s_i - s_h} \right),$$

$$\xi_j^s(L) = \begin{cases} \frac{s_i}{1 - s_i} & \text{if } L = L_{ij}, \\ \frac{1 - 2s_i}{1 - s_i}. & \text{if } L = J. \end{cases}$$

In words, $\xi_i^s(L_{ij})$ is the probability that player i chooses player $j \in J$ as a coalition partner. One can easily verify that $\sum_{j \in J} \xi_i^s(L_{ij}) = 1$ since $\sum_{j \in J} s_j = 1 - s_i$. Similarly, $\xi_j^s(L)$ is the probability that $j \in J$ chooses $L \in \underline{\mathcal{L}}_j^s$ as a coalition. Again, $\xi_j^s(L_{ij}) + \xi_j^s(J) = 1$.

Next, define a proposal rule $Y^k : \underline{\mathcal{L}}_k^s \to X$ for player $k \in I_+^s$ such that $Y^k(L)$ is player k's proposal given a coalition $L \in \underline{\mathcal{L}}_k^s$. We construct the proposal rule as follows:

- For player i: $Y_i^i(L_{ij}) = Y_j^i(L_{ij}) = 1/2$ for all $j \in J$.

- For player $j \in J$: $Y_i^j(L_{ij}) = Y_j^j(L_{ij}) = 1/2$, and

$$Y_i^j(J) = 0, \quad Y_j^j(J) = \frac{1}{2}, \quad Y_h^j(J) = s_h \left(1 + \frac{s_i + s_j - \frac{1}{2}}{1 - s_i - s_j} \right) \text{ for } h \in J \backslash \{j\}.$$

It is straightforward to check that all the proposals made under Y^k are feasible.

Given the coalition formation strategy ξ_k^s and the proposal rule Y^k, construct player k's proposal strategy $\mu_k(s)$ in state s as $\mu_k(s) = Y^k \circ \xi_k^s$, or equivalently,

$$\mu_k(y|s) = \begin{cases} \xi_k^s(L) & \text{if } y = Y^k(L) \text{ for } L \in \underline{\mathcal{L}}_k^s, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

The following lemma establishes the players' utilities in state s and that any proposal made under $\mu(s)$ is always accepted by a decisive coalition.

Lemma 4. *Fix any $s \in X_{NC}^e$. Under the proposal strategy $\mu(s)$ defined in (14),*

(i) If accepted, any proposal made under $\mu(s)$ would make the state either oligarchic or collegial in the next period.

(ii) $U_k(s) = s_k$ for all $k \in I$.

(iii) All proposals are accepted by a decisive coalition in undominated voting strategies.

(iv) $\mu_i(s)$ is an optimal proposal strategy of player i.

Proof of Lemma 4. Part (i): It is straightforward from the construction of the proposal rule Y^k.

Part (ii): Lemmas 1 and 2 imply that for any oligarchic or collegial state s, $U_k^T(s) = s_k$ for any $T \geq 1$ and $k \in I$. Then, Part (i) implies that for every proposal y offered under $\mu(s)$, $U_k^T(y) = y_k$. Then, from (2), the T-period utilities of player i and player $j \in J$ are given by

$$\begin{aligned}
U_i^T(s) &= \frac{1}{T}\left(s_i + (T-1)\sum_{k \in I} s_k \sum_{y:\mu_k(y|s)>0} \mu_k(y|s) y_i\right) \\
&= \frac{1}{T}\left(s_i + (T-1)\left(s_i \sum_{j \in J} \xi_i^s(L_{ij}) Y_i^i(L_{ij}) + \sum_{j \in J} s_j \xi_j^s(L_{ij}) Y_i^j(L_{ij})\right)\right) \\
&= \frac{1}{T}\left(s_i + (T-1)\left(\frac{s_i}{2} + \frac{s_i}{2}\right)\right) = s_i, \\
U_j^T(s) &= \frac{1}{T}\left(s_j + (T-1)\left(\frac{s_j}{2} + s_i \xi_i^s(L_{ij}) Y_j^i(L_{ij}) + \sum_{h \in J\setminus\{j\}} s_h \xi_h^s(J) Y_j^h(J)\right)\right) \\
&= \frac{1}{T}\left(s_j + (T-1)\left\{\frac{s_j}{2}\left(2 - \frac{1-2s_i}{1-s_i}\sum_{h \in J\setminus\{j\}} \frac{s_h}{1-s_i-s_h}\right) + \sum_{h \in J\setminus\{j\}} \frac{s_j s_h (1-2s_i)}{1-s_i}\left(1 + \frac{s_i + s_h - \frac{1}{2}}{1-s_i-s_h}\right)\right\}\right) \\
&= s_j.
\end{aligned}$$

Part (iii): It remains to show that all proposals are accepted by a decisive coalition in undominated voting strategies. For the proposals associated by the coalitions L_{ij} for any $j \in J$, such proposals are accepted by i and j since they receive $U_i^T(y) = U_j^T(y) = 1/2$, which is strictly greater than $U_i^T(s) = s_i$ and $U_j^T(s) = s_j$, respectively. The proposals associated with J are accepted by all members of the coalition, since the proposer j receives $U_j(y) = 1/2 > s_j = U_j(s)$ and any other player $h \in J\backslash\{j\}$ receives $U_h(y) = s_h + \frac{s_h}{1-s_i-s_j} > s_h = U_h(s)$.

Part (iv): Lemma 3 implies that the utility of a player in a noncollegial state cannot be higher than 1/2. Then, optimality follows from the fact that any proposal made under $\mu(s)$ yields 1/2 to the proposer. ■

B.2 Equilibrium Profile for $\mathbf{s} \in \mathbf{X}_{\mathbf{NC}}^{\mathbf{ne}}$

For the set X_{NC}^{ne}, as opposed to the direct construction used in Section B.1, we utilize a fixed-point theorem to construct the candidate profile.

Fix any $s \in X_{NC}^{ne}$. Let $\hat{X} = X/(X_T \cup X_D) = \{V' \in \mathbb{R}^n | \sum_i V_i' = 1, 0 \le V_i' \le \frac{1}{2}, \forall i \in I\}$. For a vector of the players' reservation values $r \in \hat{X}$ and player i's decisive coalition $L_i \in \mathcal{L}_i^s$, define player i's *proposal rule* $Y^i : \hat{X} \times \mathcal{L}_i^s \to \hat{X}$ as follows:

$$\begin{cases} Y_i^i(r; L_i) = \min\left\{\frac{1}{2}, 1 - R\right\}, & \\ Y_j^i(r; L_i) = r_j + \frac{1}{|L_i|-1} \max\left\{0, \frac{1}{2} - R\right\} & \text{for } j \in L_i\backslash\{i\}, \\ Y_k^i(r; L_i) = 0 & \text{for } k \notin L_i, \end{cases} \tag{15}$$

where $R = \sum_{h \in L_i\backslash\{i\}} r_h$. It is easy to check that $Y^i(r; L_i) \in \hat{X}$ for any $r \in \hat{X}$ and $L_i \in \mathcal{L}_i^s$. Additionally, observe that $Y_j^i(r; L_i) \ge r_j$ for any $j \in L_i$. Therefore, if r is a vector of players' reservation values in state s, $Y^i(r, L_i)$ is accepted by a decisive coalition.

Let $\Xi_i^s = \Delta\mathcal{L}_i^s$ be the set of probability distributions over $\mathcal{L}_i^s$. Let $\xi_i^s \in \Xi_i^s$ be player i's *coalition choice strategy*, where $\xi_i^s(L_i)$ is the probability that player i assigns to $L_i \in \mathcal{L}_i^s$. Given this, define player i's *best-response coalition correspondence* $B_i^s : \hat{X} \rightrightarrows \Xi_i^s$ by

$$B_i^s(r) = \arg\max_{\xi_i^s \in \Xi_i^s} \sum_{L_i \in \mathcal{L}_i^s} \xi_i^s(L_i) Y_i^i(r; L_i)$$

Since its maximand is continuous in both r and ξ_i^s, B_i^s is nonempty, compact-valued and upper hemicontinuous by the Theorem of Maximum.Additionally, B_i^s is also convex-valued for all $r \in \hat{X}$ since $Y_i^i(r; L_i) = Y_i^i(r; L_i')$ for all L_i and L_i' in the support of any $\xi_i^s \in B_i^s(r)$. Denote the profile of the best-response coalition correspondence as $B^s = \times_{i \in I} B_i^s$.

Denote the set of coalition choice strategy profiles as $\Xi^s \equiv \times_{i \in I} \Xi_i^s$. For each profile $\xi^s \in \Xi^s$ and a vector of reservation values r, define a function $\psi^s : \hat{X} \times \Xi^s \to \hat{X}$ by

$$\psi_i^s(r, \xi^s) = \sum_{j \in I} s_j \sum_{L_j \in \mathcal{L}_j^s} \xi_j^s(L_j) Y_i^j(r; L_j).$$

Note that ψ^s is nonempty and continuous in r and ξ^s.

Consider a self-map $\Psi^s : \hat{X} \rightrightarrows \hat{X}$ such that $\Psi^s(r) = \{r' \in \hat{X} | r' = \psi_i^s(r, B^s(r))\}$. Observe that $\hat{X}$ is a compact convex set, Ψ^s is upper hemicontinuous, and $\Psi^s(r)$ is a nonempty compact convex-valued set for any $r \in \hat{X}$. Therefore, Ψ^s has a fixed point by Kakutani's fixed point theorem. Denote any fixed point of Ψ^s by r^{s*}, and let $\xi^{s*} \in B^s(r^{s*})$ be such that $\psi_i^s(r^{s*}, \xi^{s*}(r^{s*})) = r^{s*}$.

Finally, we construct a candidate proposal profile $\mu(s)$ as follows:

$$\mu_i(y|s) = \begin{cases} \xi_i^{s*}(L_i) & \text{if } y = Y^i(r^{s*}; L_i) \text{ for some } L_i \in \mathcal{L}_i^s, \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

Lemma 5. *Fix any $s \in X_{NC}^{se}$. For $\mu(s)$ constructed in (16), the following are true:*

(i) If the proposal is accepted, the next period's state is either oligarchic, collegial or in X_{NC}^e.

(ii) All proposals are accepted by a decisive coalition in undominated voting strategies.

(iii) $\mu_i(s)$ is player i's optimal proposal strategy.

Proof of Lemma 5. Part (i): Without loss of generality, assume that player i is the proposer. By the construction of $\mu(s)$, player i must propose $Y^i(r^{s*}, L_i)$ for some $L_i \in \mathcal{L}_i^s$. The case with $Y_i^i(r^{s*}, L_i) = 1/2$ is straightforward, so it suffices to show that for any proposal with $Y_i^i(r^{s*}, L_i) < 1/2$, $Y^i(r^{s*}, L_i) \in X_{NC}^e$.

Let $\bar{L}_i = L_i \backslash \{i\}$ and $\underline{L}_i = I_+^s \backslash L_i$. We first prove the following claim:

Claim 4. *Suppose that $y_i = Y_i^i(r^{s*}, L_i) < 1/2$ for some proposal made by player i under $\mu(s)$. Then, $\sum_{k \in \bar{L}_i} s_k - s_j < 1/2$ for all $j \in \bar{L}_i$.*

Proof. Since $y_i < 1/2$, from (15), it must be that $y_i = 1 - \sum_{k \in \bar{L}_i} r_k^{s*}$. Suppose to the contrary that $\sum_{k \in \bar{L}_i} s_k - s_j \geq 1/2$ for some $j \in \bar{L}$. Since $s_i > 0$, it must be that $\sum_{k \in L_i} s_k - s_j > 1/2$, which implies that $L_i \backslash \{j\}$ is a decisive coalition. However, then there exists a proposal $y' = Y^i(r^{s*}, L_i \backslash \{j\})$ with

$$y'_i = \min\left\{\frac{1}{2}, 1 - \sum_{k \in \bar{L}_i \backslash \{j\}} r_k^{s*}\right\} > 1 - \sum_{k \in \bar{L}_i} r_k^{s*} = y_i$$

, which contradicts the optimality of $\mu_i(s)$. ■

To demonstrate Part (i), we claim that player i is the one who can form a coalition with every other player, i.e., $y_i + y_j > 1/2$ for any $j \in \bar{L}_i$. Suppose to the contrary that $y_i + y_j \leq 1/2$ for some $j \in \bar{L}_i$. Then, since $y_k = r_k^{s*}$ for $k \in \bar{L}_i$, $\sum_{k \in \bar{L}_i \backslash \{j\}} y_k = \sum_{k \in \bar{L}_i \backslash \{j\}} r_k^{s*} \geq 1/2$, which implies $\sum_{l \in \underline{L}} r_l^{s*} + r_j^{s*} < 1/2$. By Claim 4, $s_i + s_j + \sum_{l \in \underline{L}} s_l > 1/2$, and thus, $L' = \{i, j\} \cup \underline{L}$ is a decisive coalition. However, then there exists a proposal y'' that will be accepted by all members in L' such that $y''_i = 1/2 > y_i$, $y''_l \geq r_l^{s*}$ for $l \in L' \backslash \{i\}$ and $y''_k = 0$ for $k \notin L'$, which contradicts the optimality of $\mu_i(s)$.

Part (ii): Part (i), combined with Lemmas 1,2 and 4, implies that for any proposal y made under $\mu(s)$ with positive probability, it must be that $U_i^T(y) = y_i \geq r^{s*}$ for any $T \geq 1$ and $i \in I$. Additionally, by the construction of $Y^i(r; L_i)$, $\lim_{T \to \infty} U_i^T(s) = r_i^{s*}$ for all $i \in I$. Thus, any proposal y in the support of $\mu_i(s)$ for some $i \in I$ is approved by a decisive coalition in undominated voting strategies.

Part (iii): Assume without loss of generality that player i is the proposer. If $y_i = \frac{1}{2}$, it is obviously the optimal proposal for player i. Now assume that $y_i < \frac{1}{2}$ for some y such that $\mu_i(y|s) > 0$. Then, $y \in X_{NC}^e$ by Part (i), which implies by Lemma 4 that $U_j^T(y) = y_j$ for all $j \in I$. Additionally, $y_j = r_j^{s*,T}$ if $y_i < \frac{1}{2}$ by the construction of Y^i (equation 15). Then, y is an optimal proposal of player i, since (a) it provides no more than $U_j^T(s)$ to a set of decisive coalition partners, and (b) by the definition of B^s, y maximizes player i's utility among the proposals that satisfy (a). ■

Let us summarize our findings. Consider the following Markov profile:

- $s \in X \backslash X_{NC}$: as described in Lemmas 1 and 2.
- $s \in X_{NC}^e$: as constructed in Section B.1.
- $s \in X_{NC}^{ne}$: as constructed in Section B.2.

Then, Lemmas 4 and 5 imply that the constructed profile is an equilibrium that satisfies the conditions in Definition 1.