Patterns of Production and Trade in A Two Sector Neoclassical Growth Model: A Dynamic Model of Trade and Growth

Wontack Hong

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I. Introduction

The analyses of dynamic aspects of international trade have generally been carried out as a straightforward extention of the currently established two-sector neoclassical growth models, practically without any modifications at all.(1) The problem is that, somehow, the neoclassical growth models have concentrated exclusively on the very special case of "investment-good consumption-good" model, and hence the aspects of consumers' tastes in economic activities have been limited to the choice between the current consumption and future consumption. One may well justify this kind of

* The author is assistant professor of economics, University of Wisconsin.

(1) H. Oinki and H. Uzawa, "Patterns of Trade and Investment in a Dynamic Model of International Trade", *Review of Economic Studies*, January 1965; P. K. Bardhan, "On Factor Accumulation and the Pattern of International Specialization", *Review of Economic Studies*, January 1966 and "Equilibrium Growth in the International Economy", *Quarterly Journal of Economics*, August 1965; and R. W. Jones, "The Structure of Simple General Equilibrium Models", *The Journal of Political Economy*, December 1965.

approach in the "growth" models because their main concern is normally centered around the sequence of "savings-investment-growth". But obviously this is not the main role of consumers' tastes in the traditional trade theories. In the traditional static trade models, the consumers' choice mostly means the choice among commodities for current consumption.[2] We may say that if the traditional static trade models usually ignore the choice between current consumption and future consumption, the current neoclassical growth models as well as the dynamic trade models based on them ignore the consumers' choice among the commodities for current consumption.

The purpose of this paper is to modify the current two-sector "consumption-good investment-good" neoclassical growth model in order to obtain a dynamic trade model which does not ignore the spirit of traditionally established trade theories. We achieve the simple, and yet for most purposes very effective, generalization of the neoclassical "growth and trade" model by assuming that both labor-intensive and capital-intensive goods are used for the purposes of both investment as well as consumption.

We limit the scope of our paper by assuming Keynesian type constant average propensity to save; we abstract from the explicit savings decision and omit savings from the list of commodities. It is a kind of two-stage model in which the consumers first allocate their income between savings and consumption at a certain constant ratio, after which the classical consumption theory applies to the allocation of consumption among commodities. Couple of specific utility functions will be employed to specify the appropriate consumption models. The portion of income which is saved is used to purchase labor-intensive and capital-intensive commodities for capital formation; the ratio of those commodities purchased and used in capital formation is determined by commodity price ratio and technology of capital formation.

Furthermore, we assume the absence of complete specialization in any country. Complete specialization in any one or all countries can be handled without much difficulty; and hence we keep our paper from the exercises

(2) *Cf.* Wontack Hong, "A Global Equilibrium Pattern of Specialization: A Model to Approximate Linder's World of Production and Trade," *The Swedish Journal of Economics*, December 1969 and "The Heckscher-Ohlin Theory of Factor Price Equalization and the Indeterminacy in International Specialization", *International Economic Riview* (forthcoming).

involving complete specialization.

II. Basic Assumptions

We shall make the assumptions usual for this kind of analysis. Flows of two goods, X and Y, are respectively produced by stocks of labor (N) and homogeneous physical capital (K) in every country. Both N and K are indispensable in the production processes of X and Y. We assume perfect competition, full employment of factors, absence of transport costs, immobility of factors of production among countries, linear and homogeneous production functions which are identical for every country but different for the two industries, diminishing marginal rate of substitution of factors and that Y is always more capital-intensive than X.[3] We also assume that capital does not depreciate.

We will consider a world of country 1 and 2, country 2 representing the rest of the world as a whole from the viewpoint of country 1. We take as given the initial quantities of N and K existing in both countries, the technological knowledge and the tastes of consumers in both countries.

We first define the following set of notations.

p: the price of Y in terms of X
w: the wage-rental ratio
k: the aggregate capital-labor ratio
k_i, the capital-labor ratio in i-sector ($i=X$, Y)
n_i: the relative size of labor allocation to i-sector
x: the per capita output of commodity X
y: the per capita output of commodity Y
q_i: the per capita demand for commodity i for consumption
e_i: the per capita demand for commodity i for investment
m: the per capita imports (or exports when negative) of X
s: the average propensity to save
I: the per capita national income in terms of X

We use the "$\wedge$" for the rate of change of the variables, e.g., $\hat{x}=(1/x)(dx/dt)$. If we write $N(t)$, it implies the amount of labor supply in period

(1) A sufficient condition for Y to be always more capital-intensive than X is the identical and constant elasticity of factor substitution in the production process of X and Y.

t. We will use subscripts 1 and 2 to represent country 1 and country 2. For brevity, variables are denoted without explicitly referring to time and country when it is not too confusing by ignoring them.

III. A Static Model of International Trade

This section describes the basic structure of a simple neoclassical model of international trade on which the following analysis will be carried out.

$$x = n_x g(k_x) \tag{1}$$

$$y = n_y f(k_y) \tag{2}$$

The neoclassical hypotheses mode on the production processes are formulated in terms of production functions $g(k_x)$ and $f(k_y)$ by the following conditions:

$$\begin{gathered} g(k_x) > 0;\ g'(k_x) > 0;\ g''(k_x) < 0 \\ f(k_y) > 0;\ f'(k_y) > 0;\ g''(k_y) < 0 \\ \text{for all} \quad k_x > 0;\ k_y > 0 \end{gathered} \tag{A1}$$

$$g(0)=0;\ g(\infty)=\infty;\ f(0)=0;\ f(\infty)=\infty \tag{A2}$$

$$g'(0)=\infty;\ g'(\infty)=0;\ f'(0)=\infty;\ f'(\infty)=0 \tag{A3}$$

For any w, the optimum $k_x = k_x(w)$; $k_y = k_y(w)$ are uniquely determined by

$$w = \frac{g(k_x)}{g'(k_x)} - k_x = \frac{f((k_y)}{f'(k_y)} - k_y \tag{3}$$

Differentiating (3) with respect to w, we get

$$\begin{gathered} \frac{dk_x(w)}{dw} = \frac{-\{g'[k_x(w)]\}^2}{g[k_x(w)]\ g''[k_x(w)]} > 0 \\ \frac{dk_y(w)}{dw} = \frac{-\{f'[k_y(w)]\}^2}{f[k_y(w)]\ f''[k_y(w)]} > 0 \end{gathered} \tag{A4}$$

From the assumptions(A1)−(A3), we get

$$\begin{gathered} k_x(0)=0;\ k_y(0)=0 \\ k_x(\infty)=\infty;\ k_y(\infty)=\infty \end{gathered} \tag{A5}$$

The price of Y in terms of X is given by

$$p(w) = \frac{g'[k_x(w)]}{f'[k_y(w)]} \tag{A6}$$

and hence

$$\frac{1}{p(w)} \frac{dp(w)}{dw} = \frac{k_x - k_y}{(w+k_x)(w+k_y)} < 0 \tag{A7}$$

Therefore, $p = p(w)$ is related to the factor intensity, and the equilibrium w is uniquely determined by $p = p(w)$.

Since both capital and labor are always fully employed, we have

$$n_x+n_y=1 \tag{4}$$

$$k_x n_x+k_y n_y=k \tag{5}$$

$$I=x+py=g'(k_x)(w+k) \tag{6}$$

The eqs. (1)–(6) determine the values of the variables x, y, n_x, n_y, k_x, k_y and I for the given parameters. The parameters are the factor endowment ratio of the country k and the international commodity price p.

From eqs. (1)—(5), we may also get

$$x=\frac{k_y-k}{k_y-k_x}g(k_x) \tag{1a}$$

$$y=\frac{k-k_x}{k_y-k_x}f(k_y) \tag{2a}$$

Since the trade pattern is determined by the difference between the domestic production and domestic demand for each commodity in equilibrium, we have to introduce the demand side of the economy.

1. Demand for Consumption

Hypothesis I: Apart from price, the per capita quantity of each commodity consumed depends on per capita income, and not on the distribution of income; the consumers of a country as a whole behaving as if they were maximizing a "per capita" (or mean) utility function subject to the per capita income constraint.

Now we can express the per capita consumption demand for each commodity in each country as a function of p and I such that

$$q_i=q_i(p,\ I) \tag{A 8}$$

Each country is assumed to save a fraction of the national income (sI) and invest for capital formation. Therefore, we should have

$$(1-s)I=q_x+pq_y \tag{A 9}$$

We will assume that the average propensity to save s is identical in both countries.

2. Demand for Capital Formation

We assume that capital is a flexible machine which is a composite of X and Y. Hence, we might as well imagine an additional (assembling) industry that produces a flow of physical capital by combining X and Y without any direct use of labor or capital in the assembling process. That is,

$$e=H(e_x,\ e_y) \tag{A 10}$$

which is subject to constant returns to scale. e_i represents the per capita demand for commodity i to be used for capital formation (in the imaginary capital industry) and eN the resulting addition of physical capital to the existing captal stock K. Again, we should have

$$sI = e_x + pe_y \qquad \text{(A 11)}$$

Hypothesis II: The physical capital which is generated through the capital formation process is strictly homogeneous. The homogeneity of each unit of capital can be maintained either (1) with a fixed combination of X and Y; or (2) with variable coefficients of production which is the more neoclassical case of smooth substitutability.

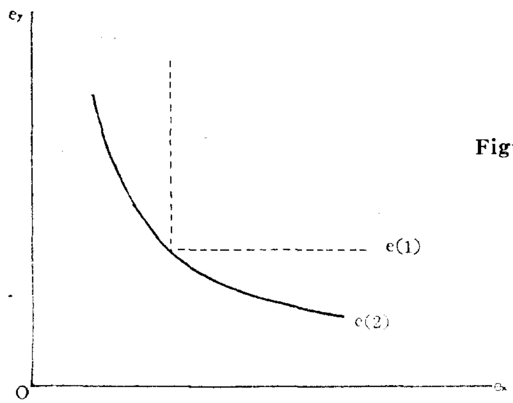

Figure 1

We can immediately see that the current "investment-good consumption-good" model is a special case of situation (1). Now we will further specify the technology of capital formation in situation (2).

With constant returns to scale, the production coefficient $\beta(=e_x/e_y)$ will be determinate function of p only. That is,

$$e = e_y h[\beta(p)] \qquad \text{(A 10 a)}$$

In the neoclassical fashion, we will assume

$$h[\beta(p)] > 0;\ h'[\beta(p)] > 0;\ h''[\beta(p)] < 0 \qquad \text{(A 12)}$$

$$h(0) = 0;\ h(\infty) = \infty \qquad \text{(A 13)}$$

$$h'(0) = \infty;\ h'(\infty) = 0 \qquad \text{(A 14)}$$

In the same fashion as $w = \frac{g}{g'} - k_x$ (3), we can have

$$p = \frac{h}{h'} - \beta \qquad \text{(A 15)}$$

If we let p_k be the price of capital in terms of X,

$$p_k e = e_y[\beta(p) + p] = sI \qquad \text{(A 16)}$$

By taking (A 10 a) and (A 15) into account, we get

$$p_k = \frac{\beta(p) + p}{h[\beta(p)]} = \frac{1}{h'} \qquad \text{(A 17)}$$

and

$$\frac{dp_k}{dp}=\frac{1}{h}>0 \qquad \text{(A 18)}$$

From (A16), we get

$$e_y=\frac{sI}{\beta(p)+p} \qquad \text{(A 19)}$$

$$e_x=\frac{\beta sI}{\beta(p)+p} \qquad \text{(A 20)}$$

3. The Reciprocal Demand Function

The structure of the short-run equilibrium quantities and prices is analyzed in terms of the reciprocal demand function. The reciprocal demand function relates the demand for the net imports of commodities by each country with the relative price prevailing in the world market. We shall derive the reciprocal demand function for the imports of X only; the demand for the imports of Y is easily obtained from the postulation that the payment of foreign trade always balances for each country.

From (1a), (6), (A 8) and (A 20), we can derive the per capita import demand for X as:

$$m=q_x(p,k)+\frac{\beta sg'(w+k)}{\beta+p}-\frac{k_y-k}{k_y-k_x}g \qquad \text{(A 21)}$$

Since the equilibrium wage-rental ratio w, the optimum capital-labor ratios k_x and k_y, and equilibrium β are determined for the given international price p, each term of eq. (A 21) and hence the per capita import demand for X are determined by the aggregate capital-labor ratio k and the international price p. Hence we denote the reciprocal demand function for X as $m(p, k)$:

$$m(p,k)=q_x(p,k)+e_x(p,k)-x(p,k) \qquad \text{(A 22)}$$

Holding k constant and under some appropriate hypotheses, the value of $m(p, k)$ increases if the international price p increases. To see this, we differentiate(A 22) with respect to p:

$$\frac{dm}{dp}=\frac{dq_x}{dp}+\frac{de_x}{dp}-\frac{dx}{dp} \qquad \text{(A 23)}$$

Now from(1a), we get

$$\frac{p}{x}\frac{dx}{dp}=\frac{p}{k_y-k_x}\left[\frac{w+k_y}{w+k_x}\frac{dk_x}{dw}+\frac{k-k_x}{k_y-k}\frac{dk_y}{dw}\right]\frac{dw}{dp} \qquad \text{(A 24)}$$

Taking (A 4) and (A 7) into account,

$$\frac{dx}{dp}<0 \qquad \text{(A 24 a)}$$

Since we assume that the per capita consumers' demand for X depends on price p and per capita income I where $I=g'(w+k)$, by differentiating (A 8) with respect to p, we get

$$\frac{p}{q_x}\frac{dq_x}{dp}=\frac{py}{I}\eta_x+\epsilon_x \qquad \text{(A 25)}$$

where $\eta_x=\frac{\partial q_x}{\partial I}\frac{I}{q_x}$(income elasticity) and $\epsilon_x=\frac{\partial q_x}{\partial p}\frac{p}{q_x}$ (income-compensated price elasticity).

Since the substitution effect ϵ_x is positive and $\partial I/\partial p=y$, if X is a normal good, we will also have positive income effect η_x and hence (A 25) will become positive also.

Hypothesis III: Neither X nor Y is an inferior good in consumption.

Remark: In the world where commodities are as broadly classified as only two categories, it would be unlikely to find an inferior good.

Now we have

$$\frac{dp_x}{dq}>0 \qquad \text{(A 25 a)}$$

By differentiating (A 20) with respect to p, we get

$$\frac{p}{e_x}\frac{de_x}{dp}=\frac{p}{\beta+p}\left\{\sigma_\beta+\frac{g'}{I}\left[\frac{(k-k_x)(w+k_y)\beta-(k_y-k)(w+k_x)p}{p(k_y-k_x)}\right]\right\}$$

$$=\frac{p}{\beta+p}(\sigma_\beta-1)+\frac{py}{I} \qquad \text{(A 26)}$$

$$=\frac{p}{\beta+p}\left[\sigma_\beta+\frac{y}{I}(\beta-\gamma)\right] \qquad \text{(A 27)}$$

where $\gamma=x/y$ and $\sigma_\beta=\frac{d\beta}{dp}\frac{p}{\beta}$

If X and Y are perfectly complementary in the process of capital formation and hence $d\beta/dp=0$, then we have

$$\frac{p}{e_x}\frac{de_x}{dp}=\frac{(\beta-\gamma)}{(\beta+p)}\frac{py}{I} \qquad \text{(A 28)}$$

where β is a constant.

Hypothesis IV: $\gamma<\beta$. That is, the ratio of labor-intensive good to capital-intensive good which are used in capital formation is greater than the ratio of labor-intensive good to capital-intensive good which are produced in the country.

Now we have

$$\frac{de_x}{dp}>0 \qquad \text{(A 27a)}$$

Remark: The sufficient condition for (A 27) to become positive is that either $\sigma_\beta > 1$ or $\gamma < \beta$. If $\sigma_\beta < 1$ and $\gamma > \beta$, then the demand for X for capital formation may increase as p falls(i.e., as the relative price of X increases.) We can understand this relationship if we examine eq.(A 20). When $\sigma_\beta > 1$, the numerator term β(the relative use of X in capital formation) decreases more than the rate of fall in p, which(though the denominator terms $\beta + p$ also decreases) is sufficient to result in the decrease in e_x. However, if $\sigma_\beta < 1$, the decrease in the numerator term β is smaller than the rate of fall in p. Furthermore, if a country produces relatively large amount of X such that $\gamma > \beta$, the fall in the other numerator term I due to the fall in p will be rather small and hence the joint term of βI will fall at much smaller rate than that of price and therefore (although the denominator terms $\beta + p$ also decreases) e_x may well increase rather than decrease with a fall in p which implies the increase in the relative price of X.

On the basis of (A 24 a), (A 25 a) and (A 27 a), we conclude that

$$\frac{dm}{dp} > 0 \qquad \text{(A 23a)}$$

Now our system of simple open economy can be closed by adding to eqs. (1)—(6) the following equation of reciprocal demand:

$$N_1 m_1(p, k_1) + N_2 m_2(p, k_2) = 0 \qquad (7)$$

With Hypotheses (I)—(IV), for any k_1 and k_2, both $m_1(p, k_1)$ and $m_2(p, k_2)$ are non-decreasing function of p, and therefore the equation of reciprocal demand uniquely determines the equilibrium international price p; which will be denoted by $p(k_1, k_2)$. System(1)—(7) expresses the miniature Walrasian general equilibrium system for a given set of (k_1, k_2).

Remark: From eqs. (1a) and (2a), we know that the smaller the k(i.e., the poorer the capital endowment) of a country, relatively more of X will be produced and hence the higher will be the ratio γ. Therefore, for a given magnitude of β and $\sigma_\beta < 1$, the capital-poor country is more likely to exhibit badly behaved offer curve compared to capital-abundant country.

Cf. Investment-Good Consumption-Good Model: The sufficient conditions for well-bahaved offer curves in the incomplete specialization are either that the investment-good is more labor-intensive in production than consumption-good or that the elasticities of factor substitution are equal to or greater than unity in both sectors in both countries.

4. Comparative Statics

We now examine the effect of the change in the relative factor endowments upon p. We will let n_1 and n_2 the relative sizes of labor force of country 1 and country 2 respectively. Since we will assume that the labor force grows at the same constant rate of $\hat{n}$ in both countries, n_1 and n_2 will be constant.

The equilibrium price $p=p(k_1, k_2)$ is obtained from the equation of reciprocal demand (7), which can now be written as:

$$n_1 m_1(p, k_1)+n_2 m_2(p, k_2)=0 \tag{7 a}$$

Differentiating (7 a) with respect to k_1 and k_2 respectively, we get

$$\frac{\partial p}{\partial k_1}=-\frac{n_1\frac{\partial m_1}{\partial k_1}}{n_1\frac{\partial m_1}{\partial p}+n_2\frac{\partial m_2}{\partial p}}$$
$$\frac{\partial p}{\partial k_2}=-\frac{n_2\frac{\partial m_2}{\partial k_2}}{n_1\frac{\partial m_1}{\partial p}+n_2\frac{\partial m_2}{\partial p}} \tag{A 29}$$

Since $\partial m/\partial k>0$ and $dm/dp>0$, we get

$$\frac{\partial p}{\partial k_1}\bigg|_{k_2=\text{constant}}<0 \tag{A 29 a}$$
$$\frac{\partial p}{\partial k_2}\bigg|_{k_1=\text{constant}}<0$$

This is a special case of the general proposition that an increase in the endowment of a particular factor of production, say capital, will raise the supply price of a commodity whose production requires relatively more labor.

IV. A Dynamic Model of International Trade

The two sector model of production and trade described by eqs. (1)—(7) can easily be used to analyze the process of global economic growth. A growth model normally allows for the growth of at least one factor of production to be determined by the system rather than given parametrically. We assume that aggregate savings form a constant percentage of the national income and that both countries have an identical and constant rate of population growth $\hat{n}$. Then the time path of capital accumulation can be described by the differential equation

$$\hat{k}(t)=\frac{e(t)}{k(t)}-\hat{n}=\frac{sI(t)}{p_k(t)k(t)}-\hat{n} \tag{8}$$

At any specific time period t, we have a given set of $[k_1(t), k_2(t)]$ and the corresponding $p(t)=p[k_1(t), k_2(t)]$. Then, according to (8), $\hat{k}(t)$ is also uniquely determined. At the next period $t+1$, we have a new set of $\{k_1(t)[1+\hat{k}_1(t)], k_2(t)[1+\hat{k}_2(t)]\}$ and the corresponding $p(t+1)=p[k_1(t+1), k_2(t+1)]$ and we can observe a certain rate of change in price $\hat{p}=(1/p)(dp/dt)$.

Let k_2 be kept constant at a certain level, and then let $k_1{}^*$ be a capital-labor ratio such that

$$sg'(k_x{}^*)\frac{k^*+w^*}{p_k{}^*k^*}=\hat{n} \tag{A 30}$$

where w^*, $p_k{}^*$ and $k_x{}^*$ are the equilibrium wage-rental ratio, equilibrium price of capital and the optimum capital-labor ratio in the X-sector respectively for the given level of equilibrium price p^* which is uniquely determined by the equation of reciprocal demand $p(k_1{}^*, k_2)$. Such a $k_1{}^*$ may be referred to as a balanced capital-labor ratio in country 1 for the given constant level of k_2.

Let us first observe that if the Hypotheses I—IV are satisfied, there always exists a uniquely determined balanced capital-labor ratio $k_1{}^*$, corresponding to each level of k_2. To see this, define

$$F_1(p)=g'(k_x)\frac{k_1(p)+w(p)}{p_k(p)k_1(p)} \tag{A 31}$$

where $k_x=k_x(p)$ and $k_1(p)$ satisfies

$$n_1m_1[p, k_1(p)]+n_2m_2(p, k_2)=0 \tag{A 32}$$

Differentiating (A 31) with respect to p to get

$$\frac{1}{F_1(p)}\frac{\partial F_1}{\partial p}=\left[\frac{\beta-\gamma_1}{\beta+p}\right]\frac{y_1}{I}-\frac{w(p)}{k_1(p)[k_1(p)+w]}\frac{\partial k_1}{\partial p} \tag{A 33}$$

which, together with $\partial k_1/\partial p<0$ (A 29a), implies that

$$\frac{\partial F_1(p)}{\partial p}>0, \text{ for all } p. \tag{A 33}$$

Therefore, for any n, there exists one and only one equilibrium price p^* for which

$$F_1(p^*)=n. \tag{A 34}$$

The corresponding capital-labor ratio $k_1{}^*=k_1(p^*, k_2)$ is a balanced capital-labor ratio for country 1, which is uniquely determined by n at the given level of k_2. $k_1(p, k_2)$ is the inverse function of $p(k_1, k_2)$.

The relation (A 33 a) and $\partial k_1/\partial p<0$ yield the following *stability theorem:* Let the Hypotheses I—IV be satisfied, and let k_2 be kept constant. Then along an arbitrary path of growth equilibrium$[K(t), N(t)]$, the capital-labor ratio $k(t)=K(t)/N(t)$ asymptotically approaches the uniquely determined balanced capital-labor ratio $k_1{}^*$. The convergence of the growth equilibrium $k_1(t)$ ismonotone; i.e., if $k_1(0)$ is greater than $k_1{}^*, k_1(t)$ decreasingly approaches $k_1{}^*$ and if $k_1(0)$ is less than $k_1{}^*$, $k_1(t)$ increasingly approaches $k_1{}^*$, and similarly all the other equilibrium variables.

Remark: The stability conditions in our dynamic model are the same as those of the static model, and the stability requires $\beta(t)>\gamma(t)$ for all t. Again, for the given magnitude of $\beta(t)$, the capital-poor country is more likely to exhibit an unstable growth path. We may explain the situation in the following fashion. The first term in eq. (8) is $sg'(w+k)/(p_k k)$. We know that $\hat{p}<0$ if $\hat{k}>0$. In such a situation, the denominator term p_k will also fall. However, since $I=g'(w+k)$ will fall more than the fall in p_k if $\beta>\gamma$, $\hat{k}$ will also decrease. However, if a capital-poor country happens to produce very large amount of x such that $\beta<\gamma$ in the country, then the fall in the per capita income I due to the fall in p will be small and (due to the fall in the denominator term p_k) $\hat{k}$ may rather increase.

In the general case where the $\beta>\gamma$ hypothesis is not necessarily satisfied, the belanced growth capital-labor ratio in country 1 for the given level of k_2 may be no longer uniquely determined, but there may be multiple balanced capital-labor ratios. However, the growth path described by (8) is globally stable; namely, the capital-labor ratio $k(t)$ satisfying (8) converges monotonically to some balanced capital-labor ratio. This is proved as follows:

In view of the *Arrow-Block-Hurwitcz theorem*, it suffices to show that the right hand side of the equation (8) tends to infinity as k goes to zero, and to zero as k goes to infinity. But k lies between $k_x(p)$ and $k_y(p)$, both of which tend to infinity (or zero). Hence, it may suffice to show the following:

$$\lim_{p\to\infty} F(p)=\infty \text{ and } \lim_{p\to 0} F(p)=0 \qquad \text{(A 35)}$$

where $F(p)$ is defined by (A 31).

The first relation in (A 35) is easily derived from(A 3), (A 4), (A 7) and (A 18). To see the second ralation, substitute (3) into (A 31) to get

$$F_1(p)=\frac{g(k_x)[w(p)+k_1(p)]}{p_k(p)k_1(p)\,[w(p)+k_x(p)]}<\frac{g(k_x)}{p_k(p)k_x}$$

which implies the second relation in (A 35).

Now we will allow for changes in k_2. Let us define

$$G_1(k_1, k_2)=\frac{sg'(w+k_1)}{p_k k_1}-\hat{n} \tag{A 36}$$

Differentiating $G_1(k_1, k_2)$ with respect to k_1, we get

$$\frac{\partial G_1}{\partial k_1}=\frac{sg'}{p_k k_1}\left[\frac{\beta-\gamma_1}{\beta+p}\frac{y_1}{g'}\frac{\partial p}{\partial k_1}-\frac{w}{k_1}\right] \tag{A 37}$$

Balanced state in country 1 implies $G_1(k_1^*, k_2)=0$ and hence

$$\left.\frac{\partial p^*}{\partial k_1^*}\right|_{G_1=0}>0 \tag{A 38}$$

and therefore, taking (A 29a) and (A 38) into account,

$$\left.\frac{dk_2}{dk_1^*}\right|_{G_1=0}<0 \tag{A 39}$$

Similarly, we get

$$\left.\frac{dk_2^*}{dk_1}\right|_{G_2=0}<\left.\frac{dk_2}{dk_1^*}\right|_{G_1=0}<0 \tag{A 40}$$

The(k_1, k_2) plane is divided into two regions according to whether $G_1(k_1, k_2)$ is positive or negative, and similarly with $G_2(k_1, k_2)$. Therefore, it is easily seen that there uniquely exists a pair (k_1^*, k_2^*) of capital-labor ratio such that

$$G_1(\bar{k}_1^*,\bar{k}_2^*)=0$$
$$G_2(\bar{k}_1^*,\bar{k}_2^*)=0 \tag{A41}$$

and since we assume the identical average propensity to save in both countries,

$$\bar{k}_1^*=\bar{k}_2^* \tag{A 42}$$

Such a pair$(\bar{k}_1^*, \bar{k}_2^*)$ may be referred to as the global balanced state; if the capital-labor ratios of the two countries are $\bar{k}_1^*$ and $\bar{k}_2^*$ then along the dynamic path of capital accumulation the capital-labor ratios both remain at the levels of $\bar{k}_1^*$ and $\bar{k}_2^*$.

The balanced state is stable, i.e., the solution $[k_1(t), k_2(t)]$ to the differential equation (8) always converges to the balanced state as time t tends to infinity. The situation is illustrated in Figure 2. The arrows in the figure describe the direction of movement of the system. We considered the case of $k_2<k_1$ only; the points above the $k_1=k_2$ line. The growth paths are described by the broken lines which indicate the ultimate convergence

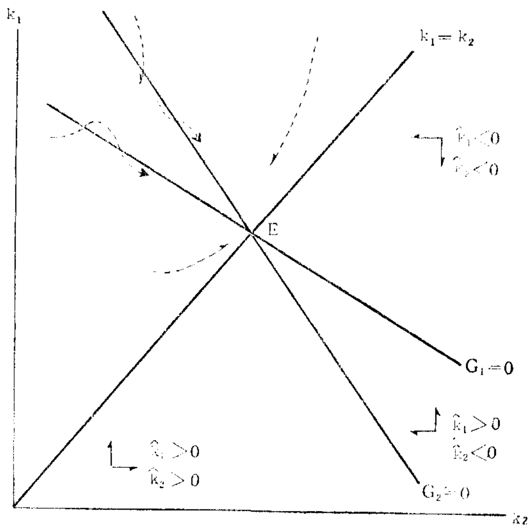

Figure 2

of the system to the global balanced state E at which $\bar{k}_1{}^*=\bar{k}_2{}^*$.

V. Rates of Changes in Price, Demand, Production and Trade

Suppose that $k_2<k_1$. Then $e_{x2}<e_{x1}$ and $x_2>x_1$ in equilibrium. Therefore, unless X is an ultranecessity which is heavily consumed in low income economy, it is likely that country 2 exports X.

Leontief Proposition: Capital-abundant country always exports capital-intensive good and labor-abundant country always exports labor-intensive good.

With the Leontief Proposition (which will be examined later introducing some specific assumptions on consumers' utility functions), we will have

$$m_1=q_{x1}+e_{x1}-x_1>0 \qquad \text{(A 43)}$$

$$m_2=q_{x2}+e_{x2}-x_2<0$$

The growth rates of production and demand for X can be expressed in terms of $\hat{k}$ and $\hat{p}$:

$$\hat{q}_x=\eta_x\left[\frac{k}{w+k}\hat{k}+\frac{py}{I}\hat{p}\right]+\epsilon_x\hat{p} \qquad (9)$$

$$\hat{e}_x = \frac{k}{w+k}\hat{k} + \left[\frac{py}{I} + \frac{p}{\beta+p}(\sigma_\beta - 1)\right]\hat{p} \tag{10}$$

$$\hat{x} = -\left\{\left[\frac{(w+k_y)^2 k_x}{(k_y-k_x)^2 w}\sigma_x + \frac{(w+k_x)(w+k_y)(k-k_x)k_y}{(k_y-k)(k_y-k_x)^2 w}\sigma_y\right]\hat{p} + \frac{k}{k_y-k}\hat{k}\right\} \tag{11}$$

where $\sigma_x = (dk_x/dw)(w/k_x)$ and $\sigma_y = (dk_y/dw)(w/k_y)$.

Suppose we start with $\hat{k}>0$ in both countries. If there were no change in price ($\hat{p}=0$), since

$$\hat{q}_x|_{\hat{q}=0}>0;\ \hat{e}_x|_{\hat{p}=0}>0;\ \hat{x}|_{\hat{p}=0}<0 \tag{A 44}$$

the import demand for X will increase in country 1 ($dm_1|_{\hat{p}=0}>0$), and the export supply of X from country 2 will decrease ($dm_2|_{\hat{p}=0}>0$). Then the commodity price p should change. If the consumption and production of X is very sensitive to changes in the relative price structure, there will be considerable adjustment of quantities demanded and produced, and hence the fall in p needed to bring about the new equilibrium in international market will be small. For a given rate of increase in k, the fall in p would generally be larger, the larger is the magnitude of η_x and the smaller are those of ϵ_x, σ_x, σ_y, σ_β and y. The larger the η_x, the larger will be the increase in consumption demand for X at $\hat{p}=0$, and hence the larger is the required price adjustment to reduce the increase in income-induced consumption. The smaller are ϵ_x, σ_β, and y, it takes a larger amount of fall in p to reduce the demand for X through the pure substitution effect and price-induced income effect. (We know that $\partial I/\partial p = y$.) And with smaller values of σ_x and σ_y, we need a larger fall in p to increase the production of X significantly enough to offset the increase in income-induced consumption of X and the decrease in production of X induced by change in factor endowment ratio. Since $\partial k_1^*/\partial p^*|_{G_1=0}>0$ (A 38), the larger the rates of fall in p, the smaller will be the difference between $k(0)$ and k^*.

1. Immizeriaing Growth

From eq. (6), we obtain the growth rate of per capita income $\hat{\imath}$:

$$\hat{\imath} = \frac{k}{w+k}\hat{k} + \frac{py}{I}\hat{p} \tag{12}$$

where $\dfrac{py}{I} = \dfrac{(w+k_y)(k-k_x)}{(w+k)(k_y-k_x)}$

Therefore, if the rate of change in p is

$$\hat{p}=\left[-\frac{k(k_y-k_x)}{(w+k_y)(k-(k_x)}\hat{k}\right]r \tag{A 45}$$

and if $r=1$, then $\hat{\imath}=0$; (12a)

if $r>1$, then $\hat{\imath}<0$; (12b)

if $r<1$, then $\hat{\imath}>0$. (12c)

We may call the situations (12a) and (12b) "Immizerizing Growth."

Suppose $\sigma_x=\sigma_y=1$. Then we have

$$\hat{x}=-\frac{k}{k_y-k}\left[1-\frac{wk+k_xk_y}{w(k-k_x)}r\right]\hat{k} \tag{11a}$$

which becomes positive if

$$r>\left[1-\frac{wk_x+k_xk_y}{wk+k_xk_y}\right]=A \tag{A46}$$

where $0<A<1$.

Furthermore, since

$$\hat{\imath}=(1-r)\frac{k}{w+k}\hat{k} \tag{12d}$$

we will have $\hat{\imath}\leqq x$, if

$$r\geqq\frac{wk-wk_x}{wk+kk_x}=B$$

Where $0<B<1$ and $A<B$. Hence, the immizeriaing growth assures that $\hat{\imath}<x$.

Since

$$\hat{p}=-\frac{k_1(k_y-k_x)\hat{k}_1r_1}{(w+k_y)(k_1-k_x)}=-\frac{k_2(k_y-k_x)\hat{k}_2r_2}{(w+k_y)(k_2-k_x)} \tag{A45a}$$

we get;

$$\frac{r_1}{B_1}\frac{B_2}{r_2}=\frac{\hat{k}_2}{\hat{k}_1} \tag{A46}$$

We know that $\hat{k}_1<\hat{k}_2$ if $k_1>k_2$. Hence, if the rate of fall in price p were such that $r_1/B_1<1$ in country 1, then B_2/r_2 should be greater than 1 and hence $r_2/B_2<1$ in country 2 also. That is, if $r_1<B_1$ and hence $x_1<\hat{\imath}_1$, then $r_2<B_2$ and hence $x_2<\hat{\imath}_2$ also. And both countries will have immizerizing growth.

2. Trade in the Balanced State

Since we assume the identical tastes and production technology in both countries, there will be no trade if $k_1=k_2$; and hence there will be no trade at the global balanced state of $(\bar{k}_1{}^*, \bar{k}_2{}^*)$. Furthermore, along the line of balanced state in each country, i.e., along the line of $G_1=0$ and $G_2=0$, the volume of trade will be changing monotonically. To see this, let us define

$$m_1[p(k_1^*, k_2), k_1^*] = m_1|_{G_1=0} \qquad \text{(A47)}$$

Differentiating $m_1\Big|_{G_1=0} = [q_{x1} + e_{x1} - x_1]_{G_1=0}$ with respect to k_1^*, we get

$$\frac{dm_1}{dk_1^*}\Big|_{G_1=0} = \left[\frac{\partial q_x}{\partial k_1^*} + \frac{\partial q_x}{\partial p}\frac{\partial p^*}{\partial k_1^*}\right]_{G_1=0} + \left[\frac{\partial e_x}{\partial k_1^*} + \frac{\partial e_x}{\partial p}\frac{\partial p^*}{\partial k_1^*}\right]_{G_1=0}$$
$$- \left[\frac{\partial x_1}{\partial k_1^*} + \frac{\partial x_1}{\partial p}\frac{\partial p^*}{\partial k_1^*}\right]_{G_1=0} \qquad \text{(A48)}$$

Since $\partial p^*/\partial k_1^*\Big|_{G_1=0} > 0$ (A 38), we have

$$\frac{dm_1}{dk_1^*}\Big|_{G_1=0} > 0 \qquad \text{(A48a)}$$

And likewise,

$$\frac{dm_2}{dk_2^*}\Big|_{G_2=0} > 0 \qquad \text{(A49)}$$

3. Leontief Proposition

A sufficient condition for the Leontief Proposition to hold is that $q_{x1} \geqq q_{x2}$ when $k_1 > k_2$. Now we will present specific consumption models in which the Leontief proposition will be assured to hold.

Hypothesis V-1: Consumer's utility function is directly additive and expenditure functions display constant marginal budget shares.

Then the utility function is the Stone-Geary form and we have linear expenditure functions:

$$u_x = \alpha u^\circ{}_x + (1-\alpha)v_x \qquad (13)$$

where the parameter α is the ratio of subsistence income to income for consumption $(1\text{-}s)I$; the other parameter v_x is the constant marginal budget share$(0 < v_x < 1;\ v_x + v_y = 1)$. Given the income for consumption $(1\text{-}s)I$ and price p, the consumer first purchases "minimum required quantities" of X and Y: q_x° of X and q_y° of Y. At the given price p, this costs $q_x^\circ + pq_y^\circ \equiv (1\text{-}s)I^\circ$ may be termed "subsistence income." He is left with $(1\text{-}s)(I\text{-}I^\circ)$ which may be termed "supernumerary income"; this he distributes among X and Y in the proportion v_x and v_y. There is of course no need to interpret the q_x° and q_y° as physical minima, but the Stone-Geary utility function basis of the linear expenditure system does require that $q_x^\circ < q_x$ and $q_y^\circ < q_y$. Here $u_x^\circ = q_x^\circ/(1\text{-}s)I^\circ$.

We now have

$$q_x = u_x(1-s)I$$
$$= q_x^\circ + (1-s)(I - I^\circ)v_x \qquad \text{(13a)}$$

Since $I_1 > I_2$ when $k_1 > k_2$, we can see that the Leontief Proposition will be assured in this linear expenditure system.

Remark: Since the trade pattern is determined by the difference between the production and demand for each commodity, without specifying the consumer's demand functions in some fashion, nothing specific can be said about the changing trade pattern. The assumption of additivity of utility function, however, reduces the scope for substitution and complementarity to the barest minimum. The neglect of 'related goods' indicates that additivity assumption appears most promising when "we are dealing with large aggregates such as clothing, food, etc. The interrelations between such aggregates probably follow largely from their competition for consumer's dollar, rather than from any more specific connection, such as exists between butter and margarine." The"directly" additive utility function excludes inferiorities, complementarities and specific substitution effects, so that it may be appropriate when goods are defined broadly as in our two-commodity model. The Stone-Geary linear expenditure system is derivable from maximization of a particular utility function which is "directly" additive. In fact, it is the only "linear" expenditure system which can be derived from a classical utility function. And this system gives some simple and systematic relationships between the income level and the magnitude of consumption as well as those of income and price elasticities. The constancy of marginal budget shares might be taken as approximation over a relevant range of I and p, that is, over the range of $k(0)$-$\bar{k}^*$ and $p(0)$-$\bar{p}^*$.

Hypothesis V-2: Consumer's utility function is quadratic and directly additive.

Then the consumer demand function becomes

$$q_x=\delta_x+\frac{1}{1+\phi p}I-\frac{1}{1+\phi p}(\delta_x+p\delta_y) \tag{14}$$

where $\delta_x=c_x/z_{xx}$; $\delta_y=c_y/z_{yy}$; $\phi=z_{xx}/z_{yy}$; and the parameters $0<c_x$, $0<c_y$; $U_x=c_x-z_{xx}q_x$; $U_y=c_y-z_{xx}q_y$; $\partial U_x/\partial q_x=-z_{xx}$; $\partial U_y/\partial q_y=-z_{yy}$; $z_{xy}=z_{yx}=0$: U_x and U_y represent the marginal utility of X and Y respectively; and the quadratic utility function is defined only where $0<c_x-z_{xx}q_x$ and $0<c_y-z_{yy}p_y$ and $q_x>0$: $q_y>0$.

Hence, we get

$$\frac{q_{x1}-q_{x2}}{1-s}=\frac{I_1-I_2}{I+\phi p}=\frac{g'(k_1-k_2)}{1+\phi p}>0 \tag{14a}$$

and the Leontief Proposition will hold.

Remark: If we assume Theil, Allen and Bowley type quadratic utility function without additivity, the existence of inferior good becomes possible. If X is an inferior good, we not only may have a badly behaving offer curves and unstable growth path, but may also have the situation of capital-abundant (high income) country exporting labor-intensive good and may end up with an unpredictable path of changes in trade pattern as the system approaches the global balanced state.

The complete set of demand functions and their properties which result from direct addilog utility function is unknown, except the trivial case of unitary income elasticities for all commodities. On the other hand. it does not appear possible to justify indirectly additive utility function by appeal to intuition, in the way that is done for direct additivity. Furthermore, the direct utility function which generates indirect "addilog" utility is generally not known explicitly.

4. Monotonic Decrease in Trade Volume

We know that if $k_1>k_2$ then $\hat{k}_1<\hat{k}_2$ and both countries become to have identical capital-labor ratio as time t becomes infinite. Since the factor endowment ratios of both countries will become closer and closer in magnitude as time passes, we may expect that the demand and production patterns of both countries will become more and more similar and hence the volume of trade will continuously diminish to become $m_1=m_2=0$ at the global balanced state. We now examine whether this "monotonic" decrease in trade volume will be assured under the Hypothesis V.

The total volume of world trade in X is

$$T=m_1+\bar{m}_2=(x_2-x_1)+(q_{x1}-q_{x2})+(e_{x1}-e_{x2}) \tag{15}$$

where $\bar{m}_2=-m_2$. Exactly the half of T is the amount of X exported or imported in eath country.

The monotonic decrease in exports and imports of X implies that $dT/dt<0$ for all t as t. And the sufficient conditions for this to happen are;

$$\frac{d\bar{x}}{dt}<0;\ \frac{d\bar{q}_x}{dt}<0;\ \frac{d\bar{e}_x}{dt}<0 \text{ for all } t, \tag{15a}$$

where $\bar{x}=x_2-x_1$; $\bar{q}_x=q_{x1}-q_{x2}$; and $\bar{e}_x=e_{x1}=e_{x2}$.

We now examine the situation with $k_1>k_2$; $\hat{k}_1>0$; $\hat{k}_2>0$.

From(A 20), we get

$$\frac{1}{\bar{e}_x}\frac{d\bar{e}_x}{dt}=\frac{p}{\beta+p}\left[\sigma_\beta+\frac{\beta(w+k_y)+p(w+k_x)}{p(k_y-k_x)}\right]\hat{p}-p(n-R)<0 \qquad \text{(A50)}$$

where $R=sg'/p_k$; $n>R$ and $\hat{p}<0$ when $\hat{k}>0$.

From(13a), we get

$$\frac{1}{q_x}\frac{d\bar{q}_x}{dt}=(1-s)v_x\left[g''\frac{dk_x}{dt}-g'(n-R)\right](k_1-k_2)<0 \qquad \text{(A51)}$$

From (14 a), we get

$$\frac{1}{\bar{q}_x}\frac{d\bar{q}_x}{dt}=(1-s)\left[\frac{w+k_y+\phi p(w+k_x)}{(k_y-k_x)(1+\phi p)}\right]\hat{p}-(n-R)<0 \qquad \text{(A52)}$$

From(1a), we get

$$\frac{1}{\bar{x}}\frac{d\bar{x}}{dt}=\frac{1}{k_y-k_x}\left\{k_y\left[\frac{k_x(w+k_y)}{k_y(w+k_x)}\sigma_x-\sigma_y\right]\frac{1}{w}\frac{dw}{dt}\right.$$
$$\left.-(n-R)(k_y-k_x)\right\}<0 \qquad \text{(A53)}$$

provided $\sigma_y \geqq \frac{wk_x+k_xk_y}{wk_y+k_xk_y}\sigma_x$ where $0<\frac{wk_x+k_xk_y}{wk_y+k_xk_y}<1$. (A53a)

Therefore, if we assume directly additive utility function which is quadratic or which displays constant marginal budget shares, the sufficient condition for monotonic decrease in trade volume is reduced simply to (A53a).

Now, taking (A 50), (A 51), and (A 52) into account and assuming (A 53a), we get

$$\frac{dT}{dt}<0 \text{ for all } t. \qquad \text{(15b)}$$

And hence, the volume of X imported (or exported) will decrease monotonically in the growth path to become zero at the global balanced state.