Nonprice Rationing Mechanism and Resource Allocation[1]

By

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I. Introduction

The purpose of this paper is to analyze consumer behavior and resource allocation in a non-Walrasian market in which price is set or fixed below the equilibrium price. In this situation nonprice rationing mechanisms such as "first come, first served," "allocation by seller's preferences," and "a centrally administered system of rationing" take over the role of price. In the textbook presentation this situation would never occur if the government did not interfere with the workings of the price system. Nonprice rationing mechanisms are, therefore, of little consequence in the study of the price system. In models of non-Walrasian markets, however, a nonmarket-clearing situation is not created by government policy but is necessarily implied by the assumptions of the models. Therefore, some nonprice rationing mechanism must be accepted as a necessary appendage to any model of non-Walrasian markets.[2] It seems, however, correct to say that these nonprice rationing mechanisms have been the subject of little economic analysis and thus little or no economic theory of them has developed.[3] I

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(2) In a non-tatonnement process the "first come, first served" principle is often used as a rationing mechanism. See Negishi [4], for example.

(3) For exceptions to this see Samuelson [6] and Tobin [7].

is the objective of this paper to show that a meaningful economic analysis of nonprice rationing mechanisms can be made. We shall, however, be dealing with only one rationing mechanism, queueing, operating under certain specified conditions.

The problem we face is why certain potential buyers of a particular commodity with excess demand would arrive at the distribution center for the commodity at a particular point in time. The arrival times of the potential customers determine the ordering of the queue at the distribution center which in turn determines who will actually be able to purchase the commodity. Thus in the case of "first come, first served," it is crucial to determine why some potential customers would arrive later than others.

We have, in a previous paper, investigated the optimal queueing decisions of individual customers under some rather stringent assumptions. In Section II of this paper we again present the model developed in [3]. In Section III we present a generalization of the queueing behavior of a potential customer developed in [3]. We analyze, in Section IV, the composition of the queue for the commodity at any point in time and also the changes in the composition of the queue as time passes. In Section V we present necessary and sufficient conditions for the formation of a black market and show that, in terms of our model, a black market does not always result in a Pareto-superior situation.

Before proceeding to the main part of this paper it should be noted that in a seminal paper Becker [1] applies his theory of the allocation of time to queues. In the sense that time is brought explicitly into the analysis of economic behavior as an element of cost, the present paper follows Becker's pioneering path. His paper, however, deals with the length of a queue in equilibrium, whereas this paper deals with the ordering of the queue, which determines the allocation of a commodity in short supply. It should also be noted that in a recent paper Nichols, Smolensky and Tideman [5] analyze waiting time in queue as a rationing mechanism.

Their paper, however, deals with the general consideration of efficie allocation constrained by equity judgement of public facilities, whereas th paper is primarily concerned with the determination of optimal arriv times, the composition of the queue, and thus the allocation of tl commodity.

II. The model

Let p denote the market price, t denote time and T denote the (fixe length of the time interval over which the commodity is offered for sale.

Assumption 1:

The sale and price of the commodity are announced together with tl time at which the sale of the commodity will begin.

We do not assume the announcement of the sale and market price ar the commencement of the sale necessarily coincide. Specifically, let t_0 deno the time of the announcement of the sale and market price and assun $t_0=0$, so that T denotes the time at the end of the time interval as we as the length of the interval. Let $\hat{t}$ denote the time at which the sale the commodity starts. Thus $t_0 \leqslant \hat{t}$, so that $\hat{t}-t_0=\hat{t}$ is the length of tin from the announcement of the sale and market price to the time tl distribution center opens, and $t-\hat{t}$, $\hat{t} \leqslant t$, is the actual selling time to time

Assumption 2:

The market price is set at the beginning of the time interval ar remains constant throughout the interval.

We let $\bar{p}$ denote the fixed price of the commodity over the time interva When a product is rationed there is usually a maximum quantity whic the consumer is allowed to purchase. Since we will be dealing with commodity in short supply, we make the following assumption;

Assumption 3:

A customer is allowed to purchase only a uniform, fixed quantity the commodity which we take to be one unit.

We now have the basic assumption;

Assumption 4:

The potential customer expects the market excess quantity demanded over the time interval to be positive and estimates the supply of the commodity to be N units.

We also assume that the number of customers arriving at the distribution center for the commodity (number of customers joining the queue for the commodity) is a random variable. Thus, on the specification of the distribution of this random variable and on the specification of the (deterministic) service time of customers, we will be able to specify the probability of a customer joining the queue at time t receiving the commodity, $P(t)$, and the expected waiting time of a customer who joins the queue at time t, $E[W(t)]$.

Since the potential customer derives utility from the commodity, we make the following assumption;

Assumption 5:

The utility of one unit of the commodity to the potential customer, denoted by U, is the maximum amount of money he is willing to give up for one unit of the commodity and is thus measurable in dollar units. Furthermore, it is assumed constant, at least over the time interval under consideration.

On joining the queue at time t the potential customer does not expect to receive the commodity with certainty. Thus he does not expect to receive utility U from joining the queue, but rather utility U discounted by the probability of receiving the commodity, i.e., expected utility. That is,

(1) $E[U]=U \cdot P(t)$.

We will disregard the costs of travel to the distribution center and the opportunity cost of the travel time. Thus for the potential customer there will be two costs involved in obtaining the commodity; the cost of the commodity (if the potential customer should receive the commodity after joining the queue) and the opportunity cost of the time spent in the queu

Since the customer can only purchase one unit of the commodity, tl direct cash cost of the commodity to a customer is p. Furthermore, tl customer, on joining the queue, does not receive the commodity wit certainty; he receives it with probability $P(t)$. Thus the expected cost (the commodity to the potential customer is $p \cdot P(t)$. Since utility was a sumed measurable in dollar units, it is meaningful to define expected coı sumer surplus as[4]

(2) $E[U-p]=[U-p]P(t)$.

In order to define the expected opportunity cost of, the time speı waiting to receive the commodity we first need the following assumptioı

Assumption 6:

The opportunity cost of time of the potential customer, denoted by , is measurable in dollar units and is assumed constant, at least ove the time interval under consideration.

Since the time the potential customer expects to spend waiting to receiv the commodity is $E[W(t)]$, the opportunity cost of time the potenti customer expects to spend waiting for the commodity is $q \cdot E[W(t)]$. Thu we can now define the potential customer's *net* expected consumer surplus a

(3) $C(t)=E[U-p]-q \cdot E[W(t)]=[U-p]P(t)-q \cdot E[W(t)]$.

The potential customer's objective is to choose his time of arrival t maximize his net expected consumer surplus, $C(t)$, subject to the conditio that his net expected consumer surplus is non-negative. If his net expecte consumer surplus is negative the potential customer will choose not to joi the queue.

III. Consumer Behavior

In our previous paper [3] on the analysis of individual consumer be

(4) Consumer surplus and hence expected consumer surplus are meaningful in our sense becaus we are not trying to define them as an area under the demand curve, but rather as th difference between the maximum amount of money the potential customer would be willin to pay for the commodity and the amount he actually does pay.

havior, we analyzed the potential customer's decision process of when and whether or not to join the queue as a two-stage decision process. That is, we recognized that the potential customer first made a decision of when to arrive at the distribution center (Stage I) and upon his arrival at the distribution center he may then also be faced with the decision of whether or not to actually join the queue for the commodity (Stage II). We showed these decisions were very similar, involving the same functional relationships with only a transformation and change of parameter. Thus for simplicity we will, in this paper, assume that the potential customer has only a one-stage decision process. That is, we will assume that he must decide if and when to arrive at the distribution center and on his arrival at the distribution center at his optimal time he joins the queue for the commodity.

In order for the potential customer to make his (single stage) decision of if and when to join the queue he must first estimate the arrival pattern (s) of other customers.

Assumption 7:

The potential customer estimates that other customers arrive at the distribution center for the commodity according to a Poisson process with parameter $\lambda(t)$, starting at time t_0.[5]

Note that a Poisson process has the property that the parameter is independent of the state of the process. That is, the rate of arrival, $\lambda(t)$, does not depend on the number of previous arrivals. Since every potential customer makes his decision of if and when to arrive at the distribution center before actually observing how many customers have arrived, the rate of arrival of customers should not depend on the number of previous arrivals.

It should also be noted that if the announcement of the sale and market

(5) Note that this is a generalization of our analysis in [3] where we assumed that the potential customer estimated customers as arriving according to a Poisson process with parameter λ.

price does not coincide with the opening of the distribution center, the assumption that the Poisson process of arriving customers starts at time t_0 means that customers may be arriving at the distribution center before sales begin.

Let

(4) $\Lambda(t)=\int_0^t \lambda(s)ds, \qquad t_0 \leqslant t \leqslant T.$

Let $X(t)$ be a random variable denoting the number of customers who have arrived for the commodity up to and including time t. Then Assumption 7 implies[6]

(5) $Pr[X(t)=j]=e^{-\Lambda(t)}\frac{[\Lambda(t)]^j}{j!}, \qquad t_0 \leqslant t \leqslant T.$

Thus the expected number of arrivals to time t is

(6) $E[X(t)]=\Lambda(t), \qquad t_0 \leqslant t \leqslant T.$

Let $D(\bar{p},t)$ be a random variable denoting the quantity of the commodity demanded at price $\bar{p}$ up to and including time t. Then since everyone arriving at the distribution center joins the queue for the commodity and since each customer can receive only one unit of the commodity we have

(7) $D(\bar{p},t)=X(t), \qquad t_0 \leqslant t \leqslant T.$

Therefore Assumption 4 and equations (6) and (7) imply

(8) $E[D(\bar{p},T)]-N=\Lambda(T)-N>0.$

Let t^0 denote the latest time the potential customer could arrive at the distribution center and upon joining the queue and expect to receive the commodity. Then t^0 is defined by

(9) $\Lambda(t^0)=\int_0^{t^0} \lambda(s)ds=N-1.$

Note that this time is *not* the last instant before the N^{th} customer arrives at the distribution center but rather the last time the potential customer could join the queue and expect to receive one full unit of the commodity. Also note Assumption 4 implies $t^0<T$.

(6) For details see Cox and Miller [2], for example.

Recall that $P(t)$ is the probability that the potential customer joining the queue at time t receives the commodity. Thus, $P(t)$ is the probability of less than N customers joining the queue to time t. That is,

$$(10)\quad P(t)=\sum_{j=0}^{N-1} Pr[D(\bar{p},t)=j]=\sum_{j=0}^{N-1} e^{-\Lambda(t)}\frac{[\Lambda(t)]^j}{j!}, \qquad t_0\leqslant t\leqslant T.$$

As stated in Section II there are two costs involved in obtaining the commodity; the expected cost of the commodity, $\bar{p}\cdot P(t)$, and the opportunity cost of the time the customer spends waiting for the commodity. Before we specify the opportunity cost of the customer's expected wait for the commodity, we first need to introduce some terminology.

Time in queue refers to the time spent in line (exclusive of service time), *service time* to the elapsed time while a customer is being served (exclusive of time in queue) and *time in the system* to time in queue plus service time.

We now turn to service time in order to determine the potential customer's expected wait in the system.

Assumption 8:

> The potential customer estimates that service takes exactly $1/\mu$ units of time per customer at the single server.(7)

Let $\hat{t}$ denote the first time the potential customer could expect to find the system empty. By Assumption 4 the potential customer expects the excess quantity demanded over the interval $[t_0,T]$ to be positive and the excess quantity demanded over the interval $[t_0,t^0]$ to be zero. Therefore the potential customer never expects to find the system empty, at least over his relevant time interval. This, then, implies $t^0<\hat{t}$.

Thus since the potential customer never expects to find the system empty over the time interval $[t_0,t^0]$, it follows that the potential customer expects the number of customers served per unit of time to be μ and expects $\mu(t-\hat{t})$ customers served to time t, $\hat{t}\leqslant t\leqslant t^0$. It also follows that, over the

(7) The assumption of a single server (single queue for the commodity) is made only for mathematical convenience and could easily be generalized without altering the analysis.

time interval $[t_0, t^0]$, the number of customers he expects to find in the system at time t is the expected number of arrivals to time t less the number served to time t, i.e.,

$$(11)\quad \begin{cases} A(t), & \text{if } t_0 \leqslant t \leqslant \hat{t}, \\ A(t)-\mu(t-\hat{t}), & \text{if } \hat{t} \leqslant t \leqslant t^0. \end{cases}$$

The length of time the potential customer, arriving at time t, for $t_0 \leqslant t \leqslant \hat{t}$, expects to wait in the system is the service time times the expected number of customers in the system at time t (including himself) plus the time until service begins. That is,

$$(12)\quad \frac{1}{\mu}[A(t)+1]+(\hat{t}-t)=\frac{A(t)}{\mu}+\frac{1}{\mu}+(\hat{t}-t), \qquad t_0 \leqslant t \leqslant \hat{t}.$$

The right hand side of equation (12) is simply the service time of the expected number in the system at time t plus the potential customer's own expected service time plus the time until service begins.

The length of time the potential customer, arriving at time t, for $\hat{t} \leqslant t \leqslant t^0$, expects to wait in the system is the service time times the expected number of customers in the system at time t (including himself). Thus we have

$$(13)\quad \frac{1}{\mu}[A(t)-\mu(t-\hat{t})+1]=\frac{A(t)}{\mu}-(t-\hat{t})+\frac{1}{\mu}, \qquad \hat{t} \leqslant t \leqslant t^0.$$

The right hand side of equation (13) is simply the service time of the expected number of customers joining the queue to time t less the elapsed time during which service has taken place plus the potential customer's own service time.

Noticing that for $t_0 \leqslant t \leqslant \hat{t}$, $-(t-\hat{t})=(\hat{t}-t)$, we may combine equations (12) and (13) into a single equation. Thus the potential customer's expected time in the system is

$$(14)\quad E[W(t)]=\frac{A(t)}{\mu}-(t-\hat{t})+\frac{1}{\mu}, \qquad t_0 \leqslant t \leqslant t^0.$$

Since q is the opportunity cost of time of the potential customer, the opportunity cost of the time the potential customer expects to wait in the system is

(15) $q \cdot E[W(t)] = q\left[\frac{\Lambda(t)}{\mu} - (t - \hat{t}) + \frac{1}{\mu}\right],$ $t_0 \leqslant t \leqslant t^0.$

The potential customer finds his optimal arrival time at the distribution center by maximizing his net expected consumer surplus (with respect to time) subject to the condition that it is non-negative. Thus the potential customer wants to find the value of t, say t^*, which maximizes

(16) $C(t) = [U - \bar{p}]P(t) - q \cdot E[W(t)] =$

$$[U - \bar{p}]\sum_{j=0}^{N-1} e^{-\Lambda(t)}\frac{[\Lambda(t)]^j}{j!} - q\left[\frac{\Lambda(t)}{\mu} - (t - \hat{t}) + \frac{1}{\mu}\right], \quad t_0 \leqslant t \leqslant t^0,$$

subject to the condition that it is non-negative. If equation (16) is negative for all t, $t_0 \leqslant t \leqslant t^0$, then the potential customer would decide not to go to the distribution center and thus not join the queue.

Theorem 1:

Necessary conditions for $C(t)$ to have a maximum at t^*, $t_0 \leqslant t^* \leqslant t^0$, are

1) $\lambda(t^*) < \mu$, and

2) $\frac{[U - \bar{p}]}{q} = \frac{E'[W(t^*)]}{P'(t^*)}$.

Sufficient conditions for $C(t)$ to have a maximum at t^*, $t_0 \leqslant t^* \leqslant t^0$, are

3) $\lambda'(t^*) \geqslant 0$, or

4) $0 > \lambda'(t^*) > [\lambda(t^*)]^2\left[1 - \frac{N-1}{\Lambda(t^*)}\right]$.

The proof of Theorem 1, being somewhat long and tedious without adding additional insight into the problem, is relegated to the Appendix.

Theorem 2:

Given that net expected consumer surplus, $C(t)$, attains a maximum over the interval $[t_0, t^0]$, a necessary and sufficient condition that the maximum is unique is that the arrival rate of customers be either monotonically increasing or decreasing over the interval, i.e., for all t, $t_0 \leqslant t \leqslant t^0$, either $\lambda'(t) > 0$ or $\lambda'(t) < 0$.

Proof:

Note that $\lambda(t) > 0$, $t_0 \leqslant t \leqslant t^0$, implies that $P'(t) < 0$ which implies that expected consumer surplus, $[U-\tilde{p}]P(t)$, is a monotonically decreasing function of time. Also note that $\lambda(t)<\mu$, $t_0 \leqslant t \leqslant t^0$, implies $E'[W(t)]<0$ which implies that the expected opportunity cost of time spent in the system, $q \cdot E[W(t)]$, is a monotonically decreasing function of time. $\lambda'(t)>0$, $t_0 \leqslant t \leqslant t^0$, if, and only if, $P''(t)<0$ and $E''[W(t)]>0$. Thus $[U-\tilde{p}]P(t)$ is concave downward and $q \cdot E[W(t)]$ is concave upward so that a maximum, if it exists, is unique. Also $\lambda'(t)<0$, $t_0 \leqslant t \leqslant t^0$, if and only if, $P''(t)>0$ and $E''[W(t)]<0$. Thus $[U-\tilde{p}]P(t)$ is concave upward and $q \cdot E[W(t)]$ is concave downward so that a maximum, if it exists, is unique.

For simplicity we will concern ourselves only with the cases where net expected consumer surplus attains a unique maximum over the time interval $[t_0, t^0]$. If we allowed more than one optimal arrival time in our analysis it would severely complicate our analysis of the composition of the queue and thus the analysis of the allocation of the commodity.

In Figures 1a and 1b, below, we show graphically expected consumer surplus and expected opportunity cost of queueing, and hence net expected consumer surplus, for the cases where $\lambda'(t)>0$ and $\lambda'(t)<0$, respectively for all t, $t_0 \leqslant t \leqslant t^0$.

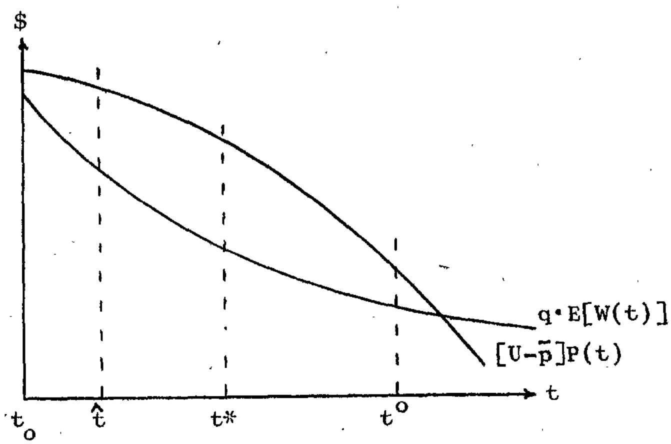

Figure 1a: $\lambda'(t)>0$

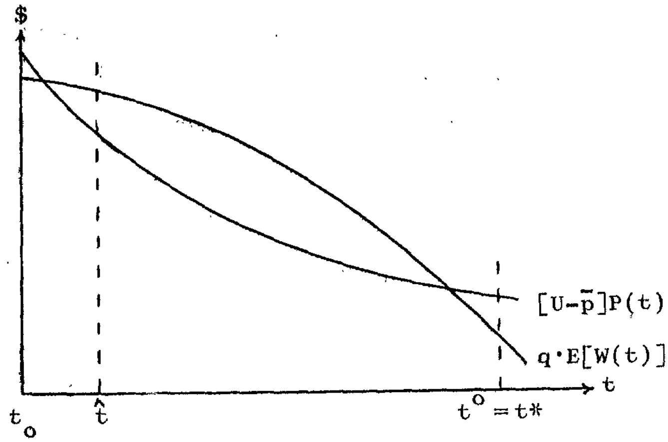

Figure 1b: $\lambda'(t)<0$

As can easily be seen from Figure 1b, for the case where $\lambda'(t)<0$, net expected consumer surplus must reach a maximum at either t_0 or t^0. Thus the optimal arrival time of the potential customer, if it exists, is either t_0 or t^0 regardless of his utility and opportunity cost of time. Hence the case where $\lambda'(t)<0$ is uninteresting. We will, therefore, consider only the case where $\lambda'(t)>0$, for all t, $t_0 \leqslant t \leqslant t^0$. This is the case where, as time passes, the rate of arrival of customers increases.

IV. Resource Allocation

As mentioned before, in order to determine the allocation of the commodity it is first necessary to determine the arrival times of all members of the population of potential customers which determine the composition of the queue at various times. Only with a heterogeneous population does it make sense to investigate the composition of the queue. We do, however, first need to abstract from the problem of customers arriving randomly because of differences in their estimation of the parameters $\lambda(t)$, μ, and N. Thus we consider the case where the population of potential customers can be differentiated only in terms of utility and opportunity cost of time.

Assumption 9:

The estimates of the parameters $\lambda(t)$, μ, and N are the same for al potential customers. Furthermore, if their decision is to join the queue they arrive at the distribution center at their optimal times.

As to the composition of the population of potential customers, we mak the follwing assumption;

Assumption 10:

Utility and opportunity cost of time vary among members of the pop lation of potential customers.

Let U_i and q_i denote, respectively, the utility and opportunity cost o time of individual members of the population of potential customers.

Theorem 3:

For all t^*, $t_0 \leqslant t^* \leqslant t^0$, $\lambda'(t^*)>0$ implies $E'[W(t^*)]/P'(t^*)$ and henc $[U_i-\bar{p}]/q_i$ vary inversely with optimal arrival times.

Proof:

From the proof of Theorem 2, $P''(t^*)<0<E''[W(t^*)]$. Also from th proof of Theorem 2, $P'(t^*)$ becomes more negative as t^* increases an $E'[W(t^*)]$ moves toward zero while remaining negative as t^* increase Thus the ratio $E'[W(t^*)]/P'(t^*)$ is positive and decreasing as t increases.

Note that Theorem 3 implies that the optimal arrival time varies inversel with utility and directly with opportunity cost of time. Thus for potenti customers with the same opportunity cost of time, Theorem 3 implies th the order of their arrival at the distribution center will be according t the magnitude of their utility for the commodity, from highest to lowes Similarly, for potential customers with the same utility for the commodit Theorem 3 implies that the order of their arrival at the distribution cent will be according to the magnitude of their opportunity cost of time, fro lowest to highest.

Furthermore, at time t_1^* those potential customers with utility and oppo tunity cost of time satisfying

$$(17)\quad \frac{[U_i-\bar{p}]}{q_i} \geqslant \frac{E'[W(t_1^*)]}{P'(t_1^*)}, \qquad t_0 \leqslant t_1^* \leqslant t^0,$$

would have joined the queue for the commodity. Similarly, from time t_1^* to t_2^* those potential customers with utility and opportunity cost of time satisfying

$$(18)\quad \frac{E'[W(t_1^*)]}{P'(t_1^*)} \geqslant \frac{[U_i-\bar{p}]}{q_i} \geqslant \frac{E'[W(t_2^*)]}{P'(t_2^*)}, \qquad t_0 \leqslant t_1^* \leqslant t_2^* \leqslant t^0,$$

would have joined the queue for the commodity.

The commodity is allocated among the population of potential customers, as long as the supply lasts, according to the ratio of consumer surplus to the opportunity cost of time. That is, the commodity will be allocated to the population of potential customers according to the magnitude of the ratio $[U_i-\bar{p}]/q_i$, in order of the largest to the smallest. Thus the commodity will be allocated first to those with relatively large utility and relatively small opportunity cost of time, then to those with either relatively large utility and relatively large opportunity cost of time or those with relatively small utility and relatively small opportunity cost of time (the order of the allocation among these two groups depending on the relative sizes of their ratio of utility and opportunity cost of time), and finally to those with relatively small utility and relatively large opportunity cost of time.

With this analysis, then, the following observation on queueing as a rationing mechanism can be made: Since everyone, rich or poor,(8) is endowed with an equal number of hours per day and the rich are usually assumed to have a higher opportunity cost of time than are the poor, relatively more of the latter are likely to receive the commodity than are the former. On the other hand, in Walrasian markets where price is the only rationing mechanism, the rich have the prerogative over the com-

(8) The terms "the rich" and "the poor" are used loosely here. Also, in using the terms "rich" and "poor" we are assuming the opportunity cost of time, q_i, varies directly with income.

modity since they can bid the highest price. Thus one may conjecture that the rich will find queueing more objectionable and distasteful than the poor.

V. Black Markets

It is almost axiomatic that where there is a rationed commodity, which is also transferable, a black market will develop. However, a black market could not develop unless people participated in it as both buyers and sellers. Ignoring any transaction costs of participating in a black market such as penalties imposed on being caught or the opportunity cost of time spent trying to find a buyer or seller, we now derive necessary and sufficient conditions under which, in the context of our model, a potential customer would participate in a black market.

Letting P be a random variable denoting possible black market prices and $p'=\bar{p}+\Delta p$, $p'\epsilon P$, we then need the following assumption;

Assumption 11:

The potential customer has a (subjective) probability distribution over black market prices, P. Furthermore, his expected value of this distribution, denoted by $E_i[P]$, is finite.

Letting σ be a random variable denoting the difference in the (fixed) market price and possible black market prices, i.e., $\Delta p\epsilon\sigma$, then that the potential customer has a (subjective) probability distribution over σ with finite expected value, denoted by $E_i[\sigma]$, is equivalent to Assumption 11.

The potential customer could participate in a black market either directly as a buyer or seller, or contract with another potential customer to buy or sell the commodity before making his decision to queue. In effect he is acting as an agent for another potential customer if he contracts to sell the commodity in the black market, whereas he is hiring an agent to wait in line for him if he contracts to buy the commodity in the black market. If the potential customer contracts to buy or sell the commodit in the black market at price $\bar{p}'(=\bar{p}+\Delta\bar{p})$, then $E_i[P]=\bar{p}'$ and $E_i[\sigma]=\Delta\bar{p}$.

A potential customer would forego joining the queue for the commodity and try to obtain the commodity in the black market at time t, $t_0 \leqslant t \leqslant T$,[9] if, and only if, his maximum net expected consumer surplus from joining the queue for the commodity (at his optimal time) was less than or equal to the consumer surplus he would expect to derive by purchasing the commodity in the black market discounted by his (estimated) probability of receiving the commodity in the black market, denoted by $V(t)$. That is,

(19) $[U_i-\bar{p}]P(t^*)-q_i \cdot E[W(t^*)] \leqslant [U_i-E_i[P]]\,V(t), \qquad t_0 \leqslant t \leqslant T.$

If the potential customer could hire someone to act as his agent (and to deliver the commodity) at price $\bar{p}'$, then $E_i[P]=\bar{p}'$. Furthermore, his (estimated) probability of receiving the commodity in the black market would be his estimate of his agent's probability of receiving the commodity on joining the queue at time t, i.e., $P(t)$. Thus the potential customer would, at time t, hire an agent to stand in line for him, if, and only if,

(20) $[U_i-\bar{p}]P(t^*)-q_i \cdot E[W(t^*)] \leqslant [U_i-\bar{p}']P(t), \qquad t_0 \leqslant t \leqslant T.$

Notice that if the potential customer's maximum net expected consumer surplus in equation (19) or (20) was negative then a necessary and sufficient condition for his participation in a black market is that his utility for the commodity be greater than or equal to the price he expects to pay in the black market.

If the potential customer was offered (with certainty) a unit of the commodity at price $\bar{p}$ in the black market, then his estimate of his probability of receiving the commodity would be one, i.e., $P(t)=1$, so that a necessary and sufficient condition for his participation in the black market would become

(21) $[U_i-\bar{p}][1-P(t^*)]+q_i \cdot E[W(t^*)] \geqslant \Delta\bar{p}.$

That is, in order for the potential customer to buy the commodity at price

(9) We do not consider the case where the potential customer joins the queue for the commodity but doesn't receive it and subsequently tries to purchase the commodity in the black market. In this case, his opportunity cost of time spent in the system would be a sunk cost and thus not relevant to the decision to purchase the commodity in the black market.

$\bar{p}'$ in the black market with certainty, the expected consumer surplus he would forego by not receiving the commodity on joining the queue at his optimal time plus his expected opportunity cost of time spent in the queue must be greater than or equal to the additional price he would have to pay in the black market. In other words, the total loss he would incur by queueing and failing to receive the commodity would have to be greater than or equal to the additional cost of the purchase in the black market.

A potential customer would join the queue at time t, $t_0 \leqslant t \leqslant T$, with the objective of reselling the commodity in the black market,(10) if, and only if, his expected consumer surplus plus expected opportunity cost of waiting in the system is less than or equal to the expected increase in price he could get by selling the commodity in the black market, discounted by his (estimated) probability of receiving the commodity if he joined the queue at time t. That is, he would join the queue if, and only if, the expected cost of the commodity to him in terms of consumer surplus foregone plus expected opportunity cost of time spent in the system is less than or equal to his expected profit from reselling the commodity in the black market. Thus we have

$$(22)\quad [U_i-\bar{p}]P(t)+q_i\cdot E[W(t)] \leqslant E_i[\sigma]P(t), \qquad t_0 \leqslant t \leqslant T.$$

If the potential customer could sell his services as an agent to another potential customer at a price $\bar{p}'$, then a necessary and sufficient condition for the potential customer to participate in the black market as an agent, at time t, is

$$(23)\quad [U_i-\bar{p}]P(t)+q_iE[W(t)] \leqslant \Delta\bar{p}\cdot P(t), \qquad t_0 \leqslant t \leqslant T.$$

From equations (19) through (21) it is clear that, other things being equal, the larger the opportunity cost of time, the more likely is the potential customer to participate in the black market as a buyer. Also

(10) We do not take up the case where the potential customer purchases the commodity for his own use and then later decides to try and sell it in the black market. Here again the opportunity cost of his waiting time in the system is a sunk cost and thus not relevant to his decision to sell the commodity in the black market.

equations (22) and (23) imply that, other things being equal, the smaller the opportunity cost of time, the more likely is the potential customer to participate in the black market as a seller.

At first glance, it would seem to follow that society would be made better off by the existence of a black market when there is a commodity rationed by time, since it is obvious that a potential customer would participate in the black market either as a buyer or seller if, and only if, he is made better off by doing so. Therefore, one is very tempted to say a situation that includes a black market is Pareto-superior to the same situation with no black market. If someone who would have queued without the intention of reselling the commodity sells it in the black market, there is no one who is deprived of the commodity and is thus made worse off as a result of this black market operation. In this case both participants are made better off and the existence of the black market results in a Pareto-superior situation. However, if someone who would not otherwise have queued for the commodity does so with the anticipation of reselling the commodity in the black market, then, because of the limited supply of the commodity, he deprives someone else of the commodity, who, if no black market existed, would have received the commodity. This forces the person who doesn't receive the commodity to do without it or to pay a higher price for it in the black market. Thus while those who participate in the black market are made better off by the existence of the black market, there are those who are made worse off. Thus, the existence of the black market does not necessarily result in a Pareto-superior situation.

It should be noted that the existence of a black market would change the arrival rate of potential customers, $\lambda(t)$, and may cause the potential customers to change their estimate of $\lambda(t)$. [11] Whether or not the existence of a black market causes the potential customers to change their

(11) Note that if the potential customers do change their estimate of $\lambda(t)$, because of Assumption 9, they all have the same revised estimate of $\lambda(t)$.

estimate of $\lambda(t)$, the above analysis of the optimal arrival times and composition of the queue remains the same (with only a possible change in the value of $\lambda(t)$). Furthermore, the above argument that a black market is not necessarily Pareto-superior to no black market is still valid. If the potential customers do not change their estimate of $\lambda(t)$ then the situation is the same as above. If, however, the potential customers do change their estimate of $\lambda(t)$, then there may still be some potential customers who do not now receive the commodity from the distribution center because there are others who join the queue with the intention of participating in the black market as a seller, even though they would not otherwise have joined the queue.

While it cannot be said that a black market necessarily brings about a Pareto-superior situation, it does result in an efficient allocation of time for the black market participants. As noted above, everything else being equal, those who participate in a black market as sellers are most likely those with low opportunity cost of time, whereas those participating as buyers are most likely those with large opportunity cost of time. Thus a black market provides a means for the relatively rich (i.e., those with large opportunity cost of time) to hire the relatively poor (i.e., those with low opportunity cost of time) to join the queue and obtain the commodity for them. The relatively rich will thus be able to have a leisurely dinner before going to a play, while the relatively poor will be able to go home a little richer and watch their almost free TV shows.

Appendix

Before preceeding with the proof of Theorem 1, we need some notational conventions. Let

$$S(t)=e^{-\Lambda(t)}\frac{[\Lambda(t)]^{N-1}}{(N-1)!},$$

and

$$R(t) = -[\lambda(t)]^2 e^{-\Lambda(t)} \frac{[\Lambda(t)]^{N-1}}{(N-1)!} + [\lambda(t)]^2 e^{-\Lambda(t)} \frac{[\Lambda(t)]^{N-2}}{(N-2)!},$$

so that

$$R(t) = S(t)[\lambda(t)]^2 \left[\frac{N-1}{\Lambda(t)} - 1\right].$$

Furthermore, by the Fundamental Theorem of the Calculus, we have that

$$\Lambda'(t) = \frac{d}{dt}\int_0^t \lambda(s)ds = \lambda(t).$$

Proof of Theorem 1:

The necessary condition that $C(t)$ have a maximum at t^*, $t_0 \leqslant t^* \leqslant t^0$, is

$$C'(t^*) = -[U-p]\lambda(t^*)S(t^*) - q\left[\frac{\lambda(t^*)}{\mu} - 1\right] = 0,$$

where $[U-p]$, $\lambda(t^*)$, $S(t^*)$, q and μ are positive. Thus $C'(t^*)=0$ if and only if, $-q[\lambda(t^*)/\mu-1]>0$ which is the case if, and only if, $\lambda(t^*)<\mu$. The value of t^* where $C'(t^*)=0$ is given by Condition 2. The sufficient condition for $C(t)$ to have a maximum at t^*, $t_0 \leqslant t^* \leqslant t^0$, is

$$C''(t^*) = -[U-p][\lambda'(t^*)S(t^*) + R(t^*)] - q\left[\frac{\lambda'(t^*)}{\mu}\right] < 0.$$

For $t_0 \leqslant t^* \leqslant t^0$, $\Lambda(t^*)/(N-1) \leqslant 1$, which implies that $R(t^*) \geq 0$. Thus if $\lambda'(t^*) \geq 0$ then

$$-[U-p][\lambda'(t^*)S(t^*) + R(t^*)] \leqslant 0,$$

and

$$q\left[\frac{\lambda'(t^*)}{\mu}\right] \geqslant 0,$$

so that $C''(t^*)<0$. If, however, $\lambda'(t^*)<0$, then $C''(t^*)<0$ if

$$-\lambda'(t^*)S(t^*) < R(t^*),$$

which is the case if, and only if,

$$\lambda'(t^*) > [\lambda(t^*)]^2 \left[1 - \frac{N-1}{\Lambda(t^*)}\right].$$

However, $C''(t^*)<0$ also if

$$-[U-p][\lambda'(t^*)S(t^*)+R(t^*)]<q\left[\frac{\lambda'(t^*)}{\mu}\right],$$

which is the case if, and only if,

$$\lambda'(t^*)>\frac{-[U-p]R(t^*)}{q/\mu+[U-p]S(t^*)}\cdot$$

However,

$$[\lambda(t^*)]^2\left[1-\frac{N-1}{\Lambda(t^*)}\right]>\frac{-[U-p]R(t^*)}{q/\mu+[U-p]S(t^*)}$$

so that all that is needed is Condition 4.

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