DERIVATION OF REGRESSION TRANSFORMATION MATRICES FOR FIRST AND HIGHER ORDER AUTOREGRESSIVE DISTURBANCES*

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I. Introduction

It is well known that if a linear regression model has disturbance terms generated by a first order autoregressive process we can transform it into a standard linear model by use of the transformation matrix.

$$(1.1)\quad T=\begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0\cdots\cdots 0 & 0 \\ -\rho & 1 & 0\cdots\cdots 0 & 0 \\ 0 & -\rho & 1\cdots\cdots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0\cdots\cdots 1 & 0 \\ 0 & 0 & 0\cdots\cdots \rho & 1 \end{bmatrix}$$

where ρ is the autoregressive coefficient. (See [3, p. 67], [4, p. 260], [5] and [7, p. 253].) The matrix T is related to the covariance matrix of disturbance terms by

$$(1.2)\quad V(u)=\sigma_0^2\ (T'T)^{-1}$$

where u is the vector of the disturbance terms, and σ_0^2 is defined later. We can easily verify that the matrix T is really a correct transformation matrix.

We can get $V(u)$ directly from the definition of autocorrelation. Because $V(u)$ is a positive definite symmetric matrix we can apply the procedure of triangular decomposition or Cholesky's method to get the inverse, $V^{-1}(u)$. (See [6, Sec. 1-4].) Because $V^{-1}(u)$ is a positive definite symmetric band matrix of width 1 (see eq. (2.20) below), the Cholesky decomposition into lower triangular matrix with positive diagonal elements gives $\sigma_0 T$ which has the same band width, 1. (See [6, Theorem 1-13].) Thus we get T from $V(u)$.

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The same procedure applies to higher order autoregression models. But the procedure of derivation of T from $V(u)$ is tedious and inefficient even for the first order case.

In this paper I will develop an alternative procedure which leads to the matrix T directly in a more efficient way. Here I will illustrate the procedure for the first and second order antoregressive models. The extension to higher order models is straightforward.

II. The First-Order Autoregression Model

Assume that observations are arranged in the order of successive time periods. Assume that the disturbances are generated by a first order autoregressive process which is described by the following equation:

(2.1) $u_t=\rho u_{t-1}+v_t,\quad t=\cdots\cdots-2,\ -1,\ 0,\ 1,\ 2,\cdots\cdots$

where

(2.2) $|\rho|<1,$

(2.3) $E(v_t)=0,$

and

(2.4) $V(v_t)=\sigma_0^2,\ \mathrm{Cov}(v_s,v_t)=0$ for $s\neq t.$

An implication derived from (2.1) is

(2.5) $u_t=v_t+\rho v_{t-1}+\rho^2 v_{t-2}+\cdots\cdots$

from which we get

(2.6) $V(u_t)=E(u_t^2)=\dfrac{\sigma_0^2}{1-\rho^2}$

and especially when $t=0$,

(2.7) $V(u_0)=\dfrac{\sigma_0^2}{1-\rho^2}.$

Now, the equation (2.1) can be rewritten as

$v_t=\rho u_{t-1}+u_t$

or in matrix form for n observations,

$$\text{(2.8)}\quad \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = \begin{bmatrix} -\rho & 1 & 0 & 0\cdots\cdots0 & 0 & 0 \\ 0 & -\rho & 1 & 0\cdots\cdots0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0\quad -\rho & 1 & 0 \\ 0 & 0 & 0 & 0\cdots\cdots0 & -\rho & 1 \end{bmatrix} \cdot \begin{bmatrix} u_0 \\ u_1 \\ \cdot \\ \cdot \\ u_n \end{bmatrix}$$

If we define

$v=(v_1\ v_2\ v_3\ \cdots\cdots v_n)'$,

$$R=\begin{bmatrix} 1 & 0\cdots\cdots & 0 & 0 \\ -\rho & 1\cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0\cdots-\rho & 1 & 0 \\ 0 & 0\cdots\cdots 0 & -\rho & 1 \end{bmatrix},\quad R^{-1}=\begin{bmatrix} 1 & 0 & 0\cdots\cdots 0 & 0 \\ \rho & 1 & 0\cdots\cdots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \rho^{n-2} & & \rho^{n-3}\cdots\cdots 1 & 0 \\ \rho^{n-1} & & \rho^{n-2}\cdots\cdots\rho & 1 \end{bmatrix}$$

$u=(u_1\ u_2\ u_3\ \cdots\cdots u_n)'$,

and

$w=v+\rho u_0 i_1$,

where i_1 is the first column of the n-th order identity matrix, then the relation (2.8) becomes

(2.9) $w=Ru.$

Because R is nonsingular we have

(2.10) $u=R^{-1}w$

(2.11) $V(u)=R^{-1}V(w)R'^{-1}.$

But because v and u_0 are uncorrelated, by (2.1) and (2.4) we have, from

(2.12) $w=v+\rho u_0 i_1$,

(2.13) $V(w)=V(v)+\rho^2 V(u_0)i_1 i_1'=\sigma_0^2 I+\dfrac{\rho^2\sigma_0^2}{1-\rho^2}i_1 i_1'$

where the second equality is due to (2.4) and (2.7). If we define a matrix

$$S=\begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0\cdots\cdots 0 \\ 0 & 1 & 0\cdots\cdots 0 \\ 0 & 0 & 1\cdots\cdots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0\cdots 0\ \ 1 \end{bmatrix},\ \text{or}\ S^{-1}=\begin{bmatrix} \frac{1}{\sqrt{1-\rho^2}} & 0 & 0\cdots\cdots 0 \\ 0 & 1 & 0\cdots\cdots 0 \\ 0 & 0 & 1\cdots\cdots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0\ \ \ 1 \end{bmatrix}$$

then the relation (2.13) becomes

(2.14) $V(w)=\sigma_0^2 S^{-2}.$

Returning to (2.11) and utilizing (2.14) we get

(2.15) $V(u)=\sigma_0^2 R^{-1}S^{-2}R'^{-1}.$

Because S or S^{-2} is a positive definite diagonal matrix we get

$$R^{-1}S^{-2}R'^{-1}=R^{-1}S^{-1}(R^{-1}S^{-1})'=(SR)^{-1}(SR)'^{-1}=(T'T)^{-1}$$

where

(2.16) $T=SR.$

Therefore we have

(2.17) $V(u)=\sigma_0^2(T'T)^{-1}$

and

(2.18) $V^{-1}(u)=\sigma_0^{-2}T'T.$

Now, let us evaluate the matrices, $T, V^{-1}(u)$ and $V(u)$. From the definitions of S and R, we get

$$(2.19)\quad T=SR=\begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0\cdots\cdots 0 & 0 \\ -\rho & 1 & 0\cdots\cdots 0 & 0 \\ 0 & -\rho & 1\cdots\cdots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0\cdots\cdots 1 & 0 \\ 0 & 0 & 0\cdots -\rho & 1 \end{bmatrix}$$

Thus we derived the transformation matrix (1.1) before we have $V(u)$.(1) Using (2.19) we evaluate

$$(2.20)\quad V^{-1}(u)=\sigma_0^{-2}T'T=\sigma_0^{-2}\begin{bmatrix} 1 & -\rho & 0 & \cdots\cdots 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots\cdots 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots\cdots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0\cdots\cdots 0 & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0\cdots\cdots 0 & -\rho & 1 \end{bmatrix}$$

If we want $V(u)$ we can evaluate it directly from (2.15) as

$$(2.21)\quad V(u)=\frac{\sigma_0^2}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 & \cdots\cdots\rho^{n-2} & \rho^{n-1} \\ \rho & 1 & \rho & \cdots\cdots\rho^{n-3} & \rho^{n-2} \\ \rho^2 & \rho & 1 & \cdots\cdots\rho^{n-4} & \rho^{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho^{n-2} & \rho^{n-3} & \rho^{n-4}\cdots\cdots\cdots & 1 & \rho \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3}\cdots\cdots\cdots & \rho & 1 \end{bmatrix}$$

(1) We sometimes have a special fourth-order autoregressive process

$u_t=\rho u_{t-4}+v_t,\ t=\cdots\cdots-2,\ -1,\ 0,\ 1,\ 2,\cdots\cdots$

for quarterly data. (See [8]). In this case the transformation matrix T_4 is obtained by simple modification of T

$T_4=S_4R_4=(S\times I_4)(R\times I_4)=SR\times I_4=T\times I_4$

where $\times$ is the Kronecker product, and S_4, R_4 are matrices S and R of the model.

III. The Second-Order Autoregression Model

Assume that the disturbances are generated by a second order autoregressive process described by the equation,

(3.1) $u_t=\rho_1 u_{t-1}+\rho_2 u_{t-2}+v_t,\ t=\cdots\cdots-2,\ -1,\ 0,\ 1,\ 2,\cdots\cdots$

where ρ_1 and ρ_2 satisfy the following constraints:

$$\rho_2+\rho_1<1$$

(3.2) $\rho_2-\rho_1<1$

$$-1<\rho_2<1.$$

Further we assume that

(3.3) $E(v_t)=0$

(3.4) $V(v_t)=\sigma_0^2,\ Cov(v_s,v_t)=0$ for $s\neq t$.

We can derive from the above that (see [2, pp. 60-62])

(3.5) $V(u_t)=\dfrac{1-\rho_2}{1+\rho_2}\ \dfrac{\sigma_0^2}{(1-\rho_2)^2-\rho_1^2}$

and

(3.6) $Cov(u_t,u_{t-1})=\dfrac{1}{1+\rho_2}\ \dfrac{\sigma_0^2\rho_1}{(1-\rho_2)^2-\rho_1^2}.$

Especially we have the covariance matrix,

(3.7) $V(u^0)=\dfrac{1}{1+\rho_2}\ \dfrac{\sigma_0^2}{(1-\rho_2)^2-\rho_1^2}\begin{bmatrix}1-\rho_2 & \rho_1\\ \rho_1 & 1-\rho_2\end{bmatrix}$

where $u^0=(u_{-1}u_0)'$.

Now equation (3.1) can be rewritten as

$$v_t=-\rho_2 u_{t-2}-\rho_1 u_{t-1}+u_t,\ t=\cdots\cdots-2,\ -1,\ 0,\ 1,\ 2,\cdots\cdots$$

or, in matrix from for n observations,

(3.8) $$\begin{bmatrix}v_1\\ v_2\\ v_3\\ \cdot\\ \cdot\\ v_n\end{bmatrix}=\begin{bmatrix}-\rho_2 & -\rho_1 & 1 & 0\cdots\cdots0 & 0\\ 0 & -\rho_2 & -\rho_1 & 1\cdots\cdots0 & 0\\ 0 & 0 & -\rho_2 & -\rho_1\cdots\cdots0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0\ -\rho_1 & 1\end{bmatrix}\begin{bmatrix}u_{-1}\\ u_0\\ u_1\\ \cdot\\ \cdot\\ u_n\end{bmatrix}$$

If we define

$$v=(v_1\ v_2\ v_3\ \cdots\cdots\ v_n)'$$

$$P=\begin{bmatrix}-\rho_2 & 0 & 0\cdots\cdots0\\ -\rho_1 & -\rho_2 & 0\cdots\cdots0\end{bmatrix}$$

$$R=\left(\begin{array}{cc:cccc}
1 & 0 & 0 & 0\cdots\cdots\ \cdots\cdots 0 & & 0\\
-\rho_1 & 1 & 0 & 0\cdots\cdots\ \cdots\cdots 0 & & 0\\
\cdots\cdots & & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & & \\
-\rho_2 & -\rho_1 & 1 & 0\cdots\cdots\ \cdots\cdots 0 & & 0\\
0 & -\rho_2 & -\rho_1 & 1\cdots\cdots\ \cdots\cdots 0 & & 0\\
\cdots\cdots & & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & & \\
\cdots\cdots & & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & & \\
0 & 0 & 0 & 0\quad -\rho_2\quad -\rho_1 & & 1
\end{array}\right)=\begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22}\end{pmatrix}$$

$$u=(u_1\ u_2\ u_3\ \cdots\cdots\ u_n)'$$

and

$$w=u+p_u^0,$$

then the relation (3.8) becomes

(3.9) $w=Ru.$

Because R is nonsingular we have

(3.10) $u=R^{-1}w.$

Therefore if we have the covariance matrix of w, $V(w)$, we can express the covariance matrix of u as

(3.11) $V(u)=R^{-1}V(w)R'^{-1}.$

But

(3.12) $w=v+Pu^0$

and v and u^0 are uncorrelated. Therefore we have

(3.13) $$V(w)=V(v)+PV(u^0)P'=\sigma_0{}^2I+\frac{\sigma_0{}^2}{1+\rho_2}\ \frac{1}{(1-\rho_2)^2-\rho_1{}^2}P\begin{bmatrix}1-\rho_2 & \rho_1\\ \rho_1 & 1-\rho_2\end{bmatrix}P'=\sigma_0{}^2Q$$

where

$$Q=\begin{bmatrix}Q_{11} & 0\\ 0 & I\end{bmatrix}$$

and

$$Q_{11}=\frac{1}{1+\rho_2}\ \frac{1}{(1-\rho_2)^2-\rho_1{}^2}\begin{bmatrix}1-\rho_2 & \rho_1\rho_2\\ \rho_1\rho_2 & (1-\rho_1{}^2)-\rho_2(1+\rho_1{}^2).\end{bmatrix}$$

The inverse of Q_{11} is

(3.14) $$Q_{11}{}^{-1}=\begin{bmatrix}(1+\rho_2)[(1-\rho_1{}^2)-\rho_2(1+\rho_1{}^2)] & -(1+\rho_2)\rho_1\rho_2\\ -(1+\rho_2)\rho_1\rho_2 & 1-\rho_2{}^2\end{bmatrix}$$

Because Q or $Q_{11}{}^{-1}$ is a positive definite symmetric matrix under condition (3.2), we can find a triangular matrix s_{11} such that

$$S_{11}=\begin{bmatrix} s_{11} & 0 \\ s_{21} & s_{22} \end{bmatrix},\quad s_{ii}>0,\quad i=1,2,$$

and

$$S_{11}'S_{11}=\begin{bmatrix} s_{11}^2+s_{21}^2 & s_{21}s_{22} \\ s_{21}s_{22} & s_{22}^2 \end{bmatrix}=Q_{11}^{-1}.$$

The result of this evaluation is

$$S_{11}=\begin{bmatrix} \sqrt{((1-\rho_2)^2-\rho_1^2)(1+\rho_2)/(1-\rho_2)} & 0 \\ -\sqrt{\rho_1^2\rho_2^2(1+\rho_2)/(1-\rho_2)} & \sqrt{1-\rho_2^2} \end{bmatrix}$$

If we define

$$S=\begin{bmatrix} S_{11} & 0 \\ 0 & I \end{bmatrix}$$

then

(3.15) $Q_{11}=S_{11}^{-1}S_{11}'^{-1}$ and $Q=S^{-1}S'^{-1}$.

Combining (3.11), (3.13) and (3.15), we get

(3.16) $V(u)=\sigma_0^2R^{-1}S^{-1}S'^{-1}R'^{-1}=\sigma_0^2(T'T)^{-1}$

where

(3.17) $T=SR$.

From (3.16) we can easily see that

(3.18) $V^{-1}(u)=\sigma_0^{-2}T'T$.

Now the evaluation of matrices T, $V^{-1}(u)$, and $V(u)$ is straightforward. From the definitions of S and R we get

$$(3.19)\quad T=SR=\begin{bmatrix} S_{11} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix}=\begin{bmatrix} S_{11}R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix}$$

$$=\left[\begin{array}{cc|cccc} s_{11} & 0 & 0 & 0\cdots & \cdots 0 & 0 \\ s_{21}-\rho_1 s_{22} & s_{22} & 0 & 0\cdots & \cdots 0 & 0 \\ \hline -\rho_2 & -\rho_1 & 1 & 0\cdots & \cdots 0 & 0 \\ 0 & -\rho_2 & -\rho_1 & 1\cdots & \cdots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0\cdots & \cdots 1 & 0 \\ 0 & 0 & 0 & 0\ -\rho_2 & -\rho_1 & 1 \end{array}\right]$$

where

$$s_{21}-\rho_1 s_{22}=-\sqrt{\rho_1^2(1+\rho_2)/(1-\rho_2)}.$$

The matrix T transforms a second order autocorrelation model

(3.20) $y=X\beta+u$

into a standard linear model

(3.21) $Ty=TX\beta+Tu$

in the sense that the covariance matrix of the disturbance vector, Tu, is a scalar matrix, i.e.,

(3.22) $V(Tu)=TV(u)T'=\sigma_0^2T(T'T)^{-1}T'=\sigma_0^2TT^{-1}T'^{-1}T'=\sigma_0^2I.$(2)

If we want $V^{-1}(u)$ and $V(u)$ we can evaluate them according to (3.18) and (3.16), respectively.

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(2) Amemiya〔1〕 suggests a simpler type of transformation matrix, $(R_{21}\ R_{22}]$. But such a transformation lacks an optımal property which T has. (See 〔5〕).