Cross-Correlogram and the Causal Structure between Two Time Series

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I. Introduction

In the social sciences, the specification of a statistical model is judged ultimately by its predictive accuracy. A model, either causal or predictive, is rejected when subsequently observed data deviate significantly from the predicted path, otherwise is maintained. No other formal test criterion for checking model specification has been used, although various techniques are available for testing the significance of parameter estimates. In this paper, we propose the use of cross-correlograms to test the causal specification of a statistical model.

The $T \times T$ correlogram of a time series is a matrix of correlation coefficients among T consecutive variates of the series. The $T \times T$ cross-correlogram between two time series is another matrix of cross-correlation coefficients between the two series of T consecutive variates respectively for an identical

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time span. By definition, the correlograms and the cross-correlograms demonstrate the basic structure of linear dependence among the variates of associated series. The primary concern of time series analysis, in both forecasting and causal studies, has been the detection of the level of dependence among the observations and the formulation of models, which show the highest level of dependence (i.e., fit well) between the "explanatory" parts and the parts to be "explained" under the given set of observations. When the model is linear, the correlograms clearly contain information necessary for this task.

There are a number of difficulties, however, when one attempts to use the correlogram to extract these information. First, the series may not be stationary to the second order, which makes it almost impossible to obtain useful estimates of the elements in the correlograms. Second, even if the series is covariance stationary, the usual estimates of covariances are often not very useful. They are additively biased (see VI below), and this additive bias is not constant across all the estimate at different lags for a given set of observations. Thus the estimated correlograms are not only biased up or down, but also exhibit non-parallel distortions. Also these estimates are not efficient in general, even when they are consistent. And finally, the correlograms are simple correlations, which reflect both direct and indirect dependences, and thereby fail to reveal the pure and direct relationship between the two variates.

Due to these difficulties, the partial correlations and the regression coefficients have been favored increasingly over the simple correlograms. The correlograms are being used only to obtain the partial correlation coefficients and spectral estimates (see Box and Jenkins [3], Jenkins and Watts [14], and Brillinger [7]).

Two different approaches, however, do make use of cross-correlograms in model specification. One quite recent approach is due to Haugh and Box [12], while the other is due to Hooker [13], and dates back to the turn of the century. Both studies were concerned with the lead-lag structure between two time series. Haugh and Box represented each time series $\{X_t(i)\}$ as a linear process, which is a linear combination of an identically and independently distributed random series $\{\varepsilon_t(i)\}$, and obtained the cross-correlogram between two series $\{\varepsilon_t(1)\}$ and $\{\varepsilon_t(2)\}$. If one or more cross-correlation coefficient is non-zero where the series $\{\varepsilon_t(1)\}$ leads $\{\varepsilon_t(2)\}$ and the cross-correlation coefficients are all zeros where $\{\varepsilon_t(2)\}$ leads $\{\varepsilon_t(1)\}$, then Haugh and Box suggest that a distributed lag regression of the series $\{X_t(2)\}$ on $\{X_t(1)\}$ is the proper specification.

Hooker, on the other hand, suggested using the cross-correlogram from the two original series to study the direction of the causal structure. He obtained the cross-correlogram between time series data on the number of marriages and the volume of retail trade in England, and attempted to infer the direction of causality therefrom. Unfortunately, his inference is based on purely intuitive arguments and lacks any formalized justifications. This old idea has been neglected until recent attention was given to it by Campbell [8]. Coen, Gomme and Kendall [9] also made use of the similar idea, but their purpose was simply choosing the most relevant causal variables for the given causal flow between the two time series. They assumed a model where the one series $\{X_t\}$ causes or at least explains the other $\{Y_t\}$, and observed only one side of the cross-correlogram where the series X leads to Y in order to select the most significant lagged variates of the series $\{X_t\}$. The present paper is an attempt to provide a formal analysis on this Hooker-Campbellian idea. Under several restrictive assumptions, it will be shown that the structure of correlogram and cross-correlogram provides information concerning the direction of causality and the lead-lag structure between two time series. The first part of this paper is devoted to the discussion of causal structure of a linear system. Then we show that under certain conditions the cross-correlogram reveals the direction of causal flow and the magnitude of causal lag for causal structures which are either unidirectional or comprise feedbacks(see A. 2 and A. 2′ in II). Some simulated results are reported to confirm our analysis. Then some extensive but inconclusive discussions are provided on the problem of estimations of the correlograms.

II. The Causal Structure and Assumptions

There have been several important definitions and discussions of causality in the current literature, Wold [19], Simon [16], Granger [11], and Sims

[17]. The causal chain of Wold is primarily concerned with the recursive or nonrecursive structure of the causal system. Wold shows that a seemingly circular causal chain often can be made nonrecursive by recognizing precise time differences. The deterministic causal system of Simon is an attempt to identify the relation between the set of exogenous variables and the set of the remaining endogenous variables, in a given system of simultaneous linear equations. The definition of causality in the works of Granger and Sims is exclusively based on forecasting accuracy. For two time series $\{X_t\}$ and $\{Y_t\}$, if a model for X_t, which incorporates past values of Y_t as explanatory variables, incurs a smaller forecast error than the model which omits the past values of Y_t, then one concludes that $\{Y_t\}$ causes $\{X_t\}$. One frequently finds this type of scheme in current econometric studies, where the minimum-forecast-error criterion is used to select the best specification among various alternatives(c.f. the distributed lag models of Jorgensen [15]). A good fitting relation, however, does not necessarily imply that the causal relationship exists in the same form. For example, it is well known that a deterministic linear relationship can provide a perfect fit to the data and yet not reflect the underlying causal relationship.

We prefer to admit that there is, testable or not in the sense of Sims, a given causal structure. One starts with assuming a type of causal structure, and then examines the data to see if it conforms to this structure. The hypothesis on the causal structure is rejected if the observations do not behave properly, and maintained otherwise. For this purpose, we need a precise definition of causality aside from the one of Granger.

A common sensical idea of causality may be formalized as a set of random variables, random functions, and binary relations. Let Y and X be random vectors such that $Y \epsilon R^M$ and $X \epsilon R^N$. Let $f(\cdot)$ and $g(\cdot)$ be random functions such that

$$f: R^M \longrightarrow R^L$$
$$g: R^N \longrightarrow R^L.$$

Let R be a binary relation in L-dimensional real space R^L. Then,

Def. 1. A causal system is a set $\{Y, X, f(\cdot), g(\cdot), R\}$ such that the random vector Y is realized for each given realization of X according to the rule

$$f(Y) R g(X). \tag{1}$$

The random vector X is said to cause Y.

Of course, the general structure of (1) is not within the scope of this paper. We will identify a most important subclass of the general causal system (1), where the function $f(\cdot)$ is linear and deterministic, the function $g(\cdot)$ is a linear and deterministic function plus a stochastic error term, and the binary relation R is the vector equality relation. The formal definition of this subclass is given below.

Def. 2. A causal system is an additively stochastic linear causal system, if Y is realized for each given realization of X by the rule

$$f_1 Y_1 + f_1 Y_2 + \cdots + f_M Y_M = g_1 X_1 + g_2 X_2 + \cdots + g_N X_N + \varepsilon, \tag{2}$$

where the f_i's and g_i's are vectors of constants and ε is a random vector.

Simon's study basically deals with the problem of identifying any arbitrary representation of a deterministic simultaneous equations system with a form (2).

We will reduce our scope further so that the random vector Y is a single variable, namely

$$Y = X'\alpha + \varepsilon. \tag{3}$$

Since we are interested in the causal structure between two time series, we will write (3) as

$$Y_t = \sum_{i=0}^{N} \alpha_i X_{t-(\tau+i)} + \varepsilon_t \tag{4}$$

where $\tau \geq 0$.

The specification of the structure (4) is not complete *per se*, and different specifications of the error term ε_t will imply different causal structures. It is well-known that different error structures require different estimation techniques for the parameters α_i's in the linear regression. On the other hand, it is important to notice how each error specification is interpreted within the context of causal structure.

The specification of error terms is fundamentally concerned with the relations between the explanatory variables(X) and the error terms (ε), and between the two error terms (ε_t and ε_{t+i}). We start with the relation between X and ε.

(a) X_t is not correlated with ε_{t+i} for all integers i (Least Squares Regresion). This specification implies that 1) there is no missing explanatory variable in (4), or if any, then they are not correlated with X_t for all t, and 2) the causal system is unidirectional from $\{X_t\}$ to $\{Y_t\}$. The first implication is obvious. As for the second one, it is sufficient to show that if there is a feedback from Y_t to X_{t+i}, then ε_t is correlated with X_{t+i}. Consider a simple case

$$Y_t = aX_{t-1} + \varepsilon_t, \quad a \neq 0,$$
$$X_t = bY_{t-1} + \eta_t, \quad b \neq 0, \qquad (4\text{-a})$$

where ε_t is not correlated with $\eta_{t'}$, for all t and t'. Notice that $\varepsilon_t(\eta_t)$ is not correlated with $X_{t-1}(Y_{t-1})$ under current specification. Then,

$$X_{t+1} = baX_{t-1} + c\varepsilon_t + \eta_{t+1},$$

and consequently

$$Cov(X_{t+1}, \varepsilon_t) = b, \quad Var(\varepsilon_t) \neq 0.$$

(b) ε_t is not correlated with X_{t-i} for any $i \geq 0$ (Contemporaneous Noncorrelation). It immediately follows from the above example that this specification allows for the feedback from $\{Y_t\}$ to $\{X_t\}$. But the feedback is not necessarily representable in the form of (4-a), namely

$$X_{t+1} = baX_{t-1} + c\varepsilon_t + \eta_{t+1}$$

where the c is any nonzero constant.

Now as for the relationship between ε_t and ε_{t+i}, we have only two possibilities such that the series $\{\varepsilon_t\}$ is either autocorrelated or not. If $\{\varepsilon_t\}$ is not autocorrelated, then the implication is straightforward. If it is autocorrelated, then either there are missing explanatory variables which are autocorrelated, or some causal flow exists from Y_t to Y_{t+i}. Particularly, if the autocorrelation coefficient of $\left\{ \sum_{i=0}^{N} \alpha_i X_{t-(r+i)} \right\}$ is the same as that of $\{\varepsilon_t\}$, then one can have a first-order autoregressive causal scheme.

Now we introduce the assumptions concerning the causal structure (4).

*A.*1 (Stationarity). The random processes $\{Y_t\}$ and $\{X_t\}$ are jointly stationary to the second order.

A set of time series are called jointly stationary to the second order, if they possess finite first and second-order moments, which are constant over time. For instance, if $\{X_t\}$ and $\{Y_t\}$ are jointly stationary to the second order, then $Cov(X_t, X_{t+\tau})=Cov(X_{t+i}, Y_{t+\tau+i})$ for all values of i and τ.

A. 2 (Unidirectionality). The random variable ε_t is not correlated with X_k for all t and k.

Assumption A. 2 is similar to the assumption made in the least squares regression model. In our case it is important to notice that A. 2 eliminates the feedback from $\{Y_t\}$ to $\{X_t\}$. If the causal flow extended from Y_t reaches, directly or indirectly, to some future values of X_t, then ε_t, which is a component of Y_t, is necessarily correlated with these future values of X_t, so assumption A. 2 breaks down. This is why we call A. 2 the assumption of unidirectionality.

Consequently, a form of feedback may be formalized as

A. 2′ (Feedback). The random variable ε_t is not correlated with X_{t-i} for all t and all $i \geq 0$.

There are several other aspects of A. 2. First, the causal model (4) must be a directly connected segment of the entire causal chain. If the causal flow from the series $\{X_t\}$ reaches $\{Y_t\}$ entirely through a third series $\{Z_t\}$, namely

$$Y_t=\sum_{i=0}^{m} a_i Z_{t-(\alpha+i)}+\varepsilon_t \tag{4-1}$$

and

$$Z_t=\sum_{i=0}^{n} b_i X_{t-(\beta+i)}+\eta_t,$$

where each equation of (4-1) satisfies A. 2. Then we have

$$\begin{aligned} Y_t &= \sum_{i=0}^{m} a_i\Big(\sum_{j=0}^{n} b_j X_{t-(\alpha+i)-(\beta+j)}+\eta_{t-(\alpha+i)}\Big)+\varepsilon_t \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} a_i b_j X_{t-(\alpha+\beta+i+j)}+\sum_{i=0}^{m} a_i \eta_{t-(\alpha+i)}+\varepsilon_t \\ &= \sum_{k=0}^{m+n} c_k X_{t-(\alpha+\beta+k)}+\xi_t. \end{aligned} \tag{4-2}$$

Assumption A. 2 does not ensure that ε_t is not correlated with X_s for all t and s in general. To solve this difficulty, we make an additional assumption, which is not unreasonable.

A. 2-1. For the causal structure (4-1), ε_t is uncorrelated with η_s for all t and s.

Since ε_t is uncorrelated with Z_s for all t and s, it follows from A. 2-1 that ε_t is also uncorrelated with X_s for all t and s, and A. 2 holds for the last expression of (4-2).

Second, the causal model (4) must have comprised the entire set of explanatory variables. If a third series $\{Z_t\}$ causes $\{Y_t\}$ simultaneously with the series $\{X_t\}$ in the form of (4), then A. 2 again fails to hold. To see this, let

$$Y_t = aX_{t-\alpha} + bZ_{t-\beta} + \eta_t, \tag{4-3}$$

where η_t is not correlated with both X_s and Z_s for all t and s. Following the expression of (4), we have $\varepsilon_t = bZ_{t-\beta} + \eta_t$ in this case. If Z_t is not uncorrelated with X_s for all t and s, then A. 2 is no more valid. Now consider an example

$$\begin{aligned} Y_t &= aX_{t-\alpha} + bZ_{t-\beta} + \eta_t, \\ Z_t &= cX_{t-\gamma} + \xi_t, \end{aligned} \tag{4-4}$$

where A. 2 holds for each equation. Then we have

$$\begin{aligned} Y_t &= aX_{t-\alpha} + b(cX_{t-\beta-\gamma} + \xi_{t-\beta}) + \eta_t \\ &= aX_{t-\alpha} + bcX_{t-\beta-\gamma} + (b\xi_{t-\beta} + \eta_t), \end{aligned} \tag{4-5}$$

in which case A. 2 clearly holds for the final expression of (4-5). Therefore, if $\{X_t\}$ is a partial cause of $\{Y_t\}$, then, in order to keep A. 2 valid for the structure (4), either $\{X_t\}$ should be entirely uncorrelated with the remaining explanatory series, or $\{X_t\}$ should also be causing the rest in the form of (4). This is a serious drawback from realism. But if the correlation between ε_t and X_s is relatively small for each pair of t and s, then our procedure still can be used descriptively.

In any case, if A. 2 does not hold, the causal model (4) is not a complete description of the causal structure between the two series, $\{X_t\}$ and $\{Y_t\}$, and another equation is needed to account for the feedback from $\{Y_t\}$ to $\{X_t\}$, or the missing links from $\{X_t\}$ to $\{Z_t\}$ and from $\{Z_t\}$ to $\{Y_t\}$, or the missing explanatory variables.

Notice that A. 2 also eliminates any kind of autoregressive causal scheme. This is not a serious problem, since the existence of autocorrelations can be accounted for directly from the causal structure (4).

III. The Correlogram and the Cross-Correlogram

The correlogram is a form of matrix arrangement of the autocorrelation coefficients of a time series. Let $\rho_{xx}(\tau)$ be the autocorrelation coefficient of the series $\{X_t\}$ for lag τ. The values of $\rho_{xx}(\tau)$ do not depend upon t, if the series $\{X_t\}$ is stationary to the second order. Clearly $\rho_{xx}(0)=1$ and $\rho_{xx}(\tau)=\rho_{xx}(-\tau)$. The $T\times T$ correlogram is the matrix

$$\begin{pmatrix} 1 & \rho_{xx}(1) & \rho_{xx}(2)\cdots\cdots\rho_{xx}(T-1) \\ \rho_{xx}(-1) & 1 & \rho_{xx}(1)\cdots\cdots\rho_{xx}(T-2) \\ \rho_{xx}(-2) & \rho_{xx}(-1) & 1\ \cdots\cdots\cdots\rho_{xx}(T-3) \\ \vdots & \vdots & \vdots \qquad\qquad \vdots \\ \rho_{xx}(-(T-1)) & \rho_{xx}(-(T-2)) & \rho_{xx}(-(T-3))\cdots\cdots 1 \end{pmatrix}$$

Similarly the cross-correlogram between two time series $\{X_t\}$ and $\{Y_t\}$ is a matrix arrangement of the cross-correlation coefficients. Let $\rho_{xy}(\tau)$ be the cross-correlation coefficient between X_t and $Y_{t+\tau}$ (i.e., X leads Y by τ). Again the values of $\rho_{xy}(\tau)$ do not depend upon t, if the series $\{X_t\}$ and $\{Y_t\}$ are jointly stationary to the second order. Notice, however, that $\rho_{xy}(0)\neq 1$ and $\rho_{xy}(\tau)\neq\rho_{xy}(-\tau)$ in general. The $T\times T$ cross-correlogram is the matrix

$$\begin{pmatrix} \rho_{xy}(0) & \rho_{xy}(1) & \rho_{xy}(2)\cdots\cdots\rho_{xy}(T-1) \\ \rho_{xy}(-1) & \rho_{xy}(0) & \rho_{xy}(1)\cdots\cdots\rho_{xy}(T-2) \\ \rho_{xy}(-2) & \rho_{xy}(-1) & \rho_{xy}(0)\cdots\cdots\rho_{xy}(T-3) \\ \vdots & \vdots & \vdots \qquad\qquad \vdots \\ \rho_{xy}(-(T-1)) & \rho_{xy}(-(T-2)) & \rho_{xy}(-(T-3))\cdots\cdots\rho_{xy}(0) \end{pmatrix}$$

Unlike the correlogram, the cross-correlogram matrix is not symmetric around $\tau=0$.

Both the correlogram and the cross-correlogram are often plotted in graphic forms. Usually the horizontal axis represents the various magnitudes of leads and the vertical axis represents the magnitude of the (cross-) correlation coefficient at each lead and lag.

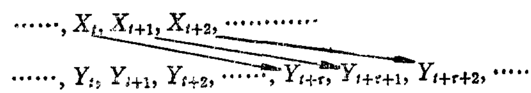
IV. The Structure of Cross-Correlogram under a Given Causal Structure

In this section, we formalize the structure of the cross-correlogram under the causal structure of (4). Then we examine the usefulness of cross-correlograms in detecting the direction of causality between the two series $\{X_t\}$ and $\{Y_t\}$.

1. We will start with the simplest case of (4), where the explanatory part consists of only a single variate, namely

$$Y_t = aX_{t-\tau} + \varepsilon_t, \tag{5}$$

where $\tau \geq 0$. Diagrammatically, the causal system (5) may be exposed as,



Let $\sigma_{xx}(i)$ indicate the covariance between X_t and X_{t+i}, and $\sigma_{xy}(i)$ the cross-covariance between X_t and Y_{t+i}. Under A. 2, the cross-covariance between $X_{t-\tau}$ and Y_t is,

$$\sigma_{xy}(\tau) = a\sigma_{xx}(0), \tag{6}$$

and the cross-covariance between X_t and Y_{t+i} for any i is

$$\begin{aligned}\sigma_{xy}(i) &= a\sigma_{xx}(i-\tau)\\ &= \rho_{xx}(i-\tau)a\sigma_{xx}(0).\end{aligned} \tag{7}$$

Therefore the cross-correlation coefficient is

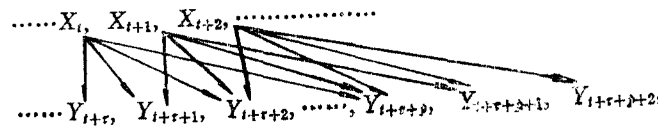
$$\begin{aligned}\rho_{xy}(i) &= \rho_{xx}(i-\tau)a\sqrt{\frac{\sigma_{xx}(0)}{\rho_{yy}(0)}}\\ &= \rho_{xx}(i-\tau)\rho_{xy}(\tau).\end{aligned} \tag{8}$$

Since $|\rho_{xx}(i-\tau)| \leq 1$ for all i, it follows that 1) the $\rho_{xy}(i)$ obtains its maximum in absolute value at $i=\tau$, which is exactly the causal lag in (5), and 2) the $\rho_{xy}(i)$ is symmetric around $i=\tau$. In this case, the cross-correlogram not only reveals the direction of the causal flow, but also the exact causal lag. And in practice, multiple maxima are very unlikely to occur, since no series is perfectly autocorrelated.

2. Now we allow the explanatory part of model (4) to contain multiple variates,

$$Y_t = \sum_{i=0}^{p} a_i X_{t-(\tau+i)} + \varepsilon_t, \tag{4}$$

where $a_i \neq 0$ for $i=0, 1, \cdots\cdots, p$ and $\tau \geq 0$. Diagrammatically, we can draw



The cross-covariance between X_t and Y_{t+k} is,

$$\sigma_{xy}(k)=\sum_{i=0}^{p}a_i\sigma_{xx}(k-\tau-i)$$

$$=\sigma_{xx}(0)(\sum_{i=0}^{p}a_i\rho_{xx}(k-\tau-i)). \qquad (9)$$

In this case the shape of the correlogram of the series $\{X_t\}$ is critical to understand the cross-correlogram. We will consider two structures for the correlogram of $\{X_t\}$.

(a) Suppose $\{X_t\}$ has a correlogram so that $\rho_{xx}(i)=\rho^{|i|}$ where $|\rho|<1$. The (stationary) first-order Markov process falls into this group. Then expression (9) becomes,

$$\sigma_{xy}(k)=\sigma_{xx}(0)(\sum_{i=0}^{p}a_i\rho^{|k-\tau-i|}). \qquad (10)$$

For $k<\tau$, $\rho^{|k-\tau-i|}=\rho^{\tau+i-k}$ and then

$$\sigma_{xy}(k)=\sigma_{xx}(0)\rho^{\tau-k}(\sum_{i=0}^{p}a_i\rho^i)$$

$$=\rho^{\tau-k}\sigma_{xy}(\tau). \qquad (11)$$

Therefore, it follows that for $k<\tau$

$$|\rho_{xy}(k)|\leq|\rho_{xy}(\tau)| \qquad (12)$$

which implies that the maximum(in absolute value) of the cross-correlogram occurs somewhere to the right of lag τ.

If $k=\tau+p+j$ with $j>0$, then

$$\rho^{|k-\tau-i|}=\rho^j\rho^{p-i}$$

and

$$\sigma_{xy}(k)=\rho^j\sigma_{xx}(0)(\sum_{i=0}^{p}a_i\rho^{p-i})$$

$$=\rho^j\sigma_{xx}(\tau+p). \qquad (13)$$

Consequently, as k increases beyond $\tau+p$, the absolute value of $\sigma_{xy}(k)$ decreases monotonically. Therefore the maximum occurs between lags τ and $\tau+p$. If p goes to infinity, then $\sigma_{xy}(k)$ may never diminish as k gets larger and larger. In addition symmetry cannot be established in general.

(b) Suppose the series $\{X_t\}$ possesses a correlogram such that $\rho_{xx}(i) \geq 0$ for all i and $\rho_{xx}(i) \geq \rho_{xx}(j)$ whenever $|i| \leq |j|$. Also assume that the a_i's have the same sign. Then for $k_1 < k_2 < \tau$, it follows that

$$\rho_{xx}(k_1-\tau-i) \leq \rho_{xx}(k_2-\tau-i) \tag{14}$$

for all $i=0, 1, \cdots, p$. Since the signs for the a_i's are identical, it follows from (9) that

$$|\sigma_{xy}(k_1)| \leq |\sigma_{xy}(k_2)|. \tag{15}$$

The absolute value of the cross-correlogram decreases as k gets smaller for $k < \tau$. For $k_1 > k_2 > \tau + p$, we have

$$\rho_{xx}(k_1-\tau-i) \leq \rho_{xx}(k_2-\tau-i) \tag{16}$$

and similarly

$$|\sigma_{xy}(k_1)| \leq |\sigma_{xy}(k_2)| \tag{17}$$

which implies that the cross-correlogram diminishes as k gets greater for $k > \tau + p$. Again no symmetry can be obtained.

In summary, if either 1) the series $\{X_t\}$ is a first-order Markov process, or 2) it has a non-increasing, non-negative correlogram and the signs of causal weights (a_i's) are identical, then the location of the maximum of the correlogram reveals the direction of causal flow between the two series.

3. *Feedback*. A simple linear feedback may be formalized as

$$\begin{aligned} Y_t &= aX_{t-\alpha} + \varepsilon_t, \\ X_t &= bY_{t-\beta} + \eta_t, \end{aligned} \tag{4-6}$$

where $|ab| < 1$ is required for the stationarity. Then under A. 2′ the cross-covariance for $i \geq 0$ is

$$\begin{aligned} \sigma_{xy}(i) &= a\sigma_{xx}(\alpha - i), \\ \sigma_{xy}(-i) &= b\sigma_{yy}(\beta - i), \\ \text{and} \quad a\sigma_{xx}(\alpha) &= b\sigma_{yy}(\beta). \end{aligned} \tag{4-7}$$

A distributed lag linear feedback may be formalized as

$$\begin{aligned} Y_t &= \sum_{k=0}^{N} a_k X_{t-(\alpha+k)} + \varepsilon_t, \\ X_t &= \sum_{j=0}^{M} b_j Y_{t-(\beta+j)} + \eta_t, \end{aligned} \tag{4-8}$$

where the constants a_k's and b_j's establish the joint stationarity of the two series $\{X_t\}$ and $\{Y_t\}$. Then the cross-covariance for $i \geq 0$ is

$$\sigma_{xy}(i)=\sum_{k=0}^{N} a_k \sigma_{xx}(\alpha+k-i),$$

$$\sigma_{xy}(-i)=\sum_{j=0}^{M} b_j \sigma_{yy}(\beta+j-i),$$

and $$\sum_{k=0}^{N} a_k \sigma_{xx}(\alpha+k)=\sum_{j=0}^{M} b_j \sigma_{xx}(\beta+j). \quad (4\text{-}9)$$

So the analyses in sections 1 and 2 are still valid for each side of the cross-correlogram, and consequently, the cross-correlogram reveals the structure of feedback under the same conditions as sections 1 and 2.

4. *Some Pitfalls in Interpretation.* Cross-correlograms which possess the same shapes as those discussed above can be obtained even though the two series have no direct causal relationship. Consider a causal structure

$$\begin{aligned} Y_t &= aZ_{t-\tau}+\varepsilon_t \\ X_t &= bZ+\eta_t \end{aligned} \quad (18)$$

where each equation satisfied A.2 and ε_t is not correlated with η_s for all t and s. Then the cross-covariance between X_t and Y_{t+i} is

$$\sigma_{xy}(i)=ab\sigma_{zz}(i-\tau). \quad (19)$$

Consequently, the cross-correlogram will assume its maximum in absolute value at $i=\tau$, and be symmetrical around $i=\tau$, which might mislead one to hypothesize a causal flow from $X_{t-\tau}$ to Y_t. By examining the cross-correlogram between $\{X_t\}$ and $\{Y_t\}$, there is no way to distinguish the causal structure (5) from (18). In such a situation, causal structure (5) would not be rejected.

Despite the several limitations, however, we can use the cross-correlogram to reject a certain causal hypothesis, if the shape of the cross-correlogram is not consistent with the hypothesized causal structure.

V. Some Simulated and Practical Results

In view of the above analyses, numerous simulated works are performed and only some of them, which are typical, are reported in this section.

Despite the controversial difficulties discussed in VI concerning the statistical properties of empirical correlograms, every result of our simulation confirms the conclusions of the above analyses. The following list summarizes the causal structures between each pair of time series, of which the graphic cross-correlograms are reported.

Fig. 1. Unidirectional structure with a single causal variable and spherical error.

$X_t = -0.7Y_{t-5} + \varepsilon_t, \quad \rho_{yy} = 0.8.$

Fig. 2. Unidirectional structure with a single causal variable and autocorrelated error.

$Y_t = -2.5X_{t-2} + \varepsilon_t, \quad \rho_{xx} = 0.8, \quad \rho_{\varepsilon\varepsilon} = 0.7.$

Fig. 3. Unidirectional structure with multiple causal variables and spherical error (1).

$Y_t = 1.7X_{t-2} + 1.2X_{t-3} + 1.0X_{t-4} + 0.7X_{t-5} + 0.2X_{t-6} + \varepsilon_t,$

(X_t is a Markov process with $\rho_{xx} = 0.8$).

Fig. 4. Unidirectional structure with multiple causal variables and spherical error (2).

$Y_t = 0.7X_{t-2} - 1.2X_{t-3} + 1.7X_{t-4} - 1.0X_{t-5} + 0.2X_{t-6} + \varepsilon_t.$

(X_t is a Markov process with $\rho_{xx} = 0.8$).

Fig. 5. Unidirectional structure with multiple causal variables and autocorrelated error.

$Y_t = 1.7X_{t-1} + 1.1X_{t-2} + \varepsilon_t, \quad \rho_{xx} = 0.8, \quad \rho_{\varepsilon\varepsilon} = 0.7.$

Fig. 6. Negative linear feedback.

$X_t = 0.8Y_{t-2} + \varepsilon_t, \quad Y_t = -0.7X_{t-2} + \eta_t, \quad \rho_{\varepsilon\varepsilon} = 0.7, \quad \rho_{\eta\eta} = 0.8.$

Fig. 7. Positive linear feedback.

$X_t = 0.3Y_{t-1} + \varepsilon_t, \quad Y_t = 2.7X_{t-1} + \eta_t.$

Fig. 8. Non-proportional positive feedback.

$X_t = 1.7Y_t + \varepsilon_t, \quad Y_t = 2.1\varepsilon_{t-1} + \eta_t.$

Fig. 9. Non-proportional negative feedback.

$$X_t=1.7Y_t+\varepsilon_t,\quad Y_t=-2.1\varepsilon_{t-1}+\eta_t.$$

The figures 8 and 9 demonstrate the asymetric cross-correlograms such that the shoulder of the right hand side is higher(lower) for Fig. 8 (Fig.9). One can see for Fig. 8,

$$\sigma_{xy}(-1)=1.7\sigma_{yy}(1),$$
$$\sigma_{xy}(1)=1.7\sigma_{yy}(1)+2.1\sigma_\varepsilon^2.$$

The following figures (Fig. 10-16) are obtained from the miscellaneous practical data. For each figure, a tentative inference about the causal structure is attempted.

Fig. 10. X=annual rate of total admissions to mental hospital, U.S. male (1914-1960)
Y=annual rate of the U.S. employment (1914-1960)
(source: the graphs in Brenner[5], pp. 76).

Fig. 11. X=annual rate of total admissions to mental hospital, U.S. female (1914-1960)
Y=annual rate of the U.S. employment (1914-1960)
(source: the graphs in Brenner[5], pp. 76).

From Fig. 10 one may weakly conclude that "lower employment causes a higher male total admissions to mental hospital after 1～3 years." But the increasing tendency on the right shoulder is likely to indicate a different shape of the cross-correlogram, if we had more observations. Fig. 11 suggests a very unlikely hypothesis, "high female admissions to mental hospital causes a high employment after 3 years or more." In this case, it would rather be safe to interpret as a round-about causal relation or a simple coincidence (see (4-1), (4-4) and (18)).

Fig. 12. X=annual rate of the cirrhosis mortality, U.S. white male (1934-1968)
Y=1 year lagged U.S. average alcohol consumption (1934-68)
(source: the graphs in Brenner[6]).

Fig. 13. X=annual rate of the cirrhosis mortality, U.S. white female (1934-1968)
Y=1 year lagged U.S. average alcohol consumption (1934-68)

(source: the graphs in Brenner[6]).

Both of Fig. 12 and 13 suggest that "high level of the alcohol consumption results in a high cirrhosis mortality after 1 year," which is consistent with the conclusion of Brenner[6]. Furthermore, both figures indicate non-proportional negative feedback from the cirrhosis mortality to the average alcohol consumption, i.e., both figures exhibit lower right hand side shoulders (see Fig. 9).

Fig. 13-1. X=annual rate of murders (1933-1973)
Y=probability of execution (1933-1973)
(see Ehrlich[10] for the source of data).

The degree of linear relationship is negligible in Fig. 13-1. Nonetheless it seems to indicate the existence of slight feedback between the series. An increase in murders result in the higher probability of executions for the subsequent 1 through 11 years, and an increase of probability of execution seems to reduce the murder rate after 6 through 11 years. But the cross-correlation coefficients are too small to draw any significant conclusions.

Fig. 14. X=annual rate of the U.S. GNP (1947-1974)
Y=annual rate of the U.S. money supply excluding the time deposits (1947-1974)
(source: Survey of Current Business, 1976, biennnial edition).

Fig. 15. X=quarterly rate of the U.S. GNP (1947-1974)
Y=quarterly rate of the U.S. money supply excluding the time deposits (1947-1974)
(source: Survey of Current Business, 1976, biennial edition).

Fig. 16. X=quarterly rate of the U.S. GNP (1947-1974)
Y=quarterly rate of the U.S. money supply including the time deposits (1947-1974)
(source: Survey of Current Business, 1976, biennial edition).

All of the above three figures exhibit very high peak at the lag zero, and higher (uniformly) shoulder on the side where the money supply leads to the gross national products. If we hypothesize that current money supply causes the current GNP, then the non-proportional negative feedback from

GNP to the money supply is indicated, which is not realistic. On the other hand, if we hypothesize that the current GNP causes the current money supply, then the non-proportional feedback is positive from the money supply to the GNP of the subsequent periods. Notice that this (latter) hypothesis does not coincide with the conclusion in Sims [17]. But the asymmetry is not very distinguishable and therefore, one can only safely conclude that if there is any causal flow between the money supply and the gross national products, then the significant lag must be within a quarter.

VI. The Estimation of Covariances

The practical use of the estimated auto- and cross-covariances has been discouraged mainly because the widely used estimators exhibit often poor statistical properties. In general a time series is called ergodic, if the estimates of its moments of all orders are consistent. The ergodicity depends heavily upon the structure of moments of the series. An example of the ergodic series is the linear process, which is assumed in the works of Haugh and Box [12]. The linear process is a random process which can be represented by a linear combination of the white noise series where the sum of the squared weights is finite. Such a series is always ergodic.

In this section, we discuss the general structure of bias and the asymptotic property of the covariance estimates for two jointly covariance stationary time series, and examine their applicability for the methods analyzed in previous sections.

1. *The Empirical Mean.* The usual estimator on the mean for the finite stretch of the series $\{X_t\}$ is such that

$$\bar{X}=\frac{1}{T}\sum_{t=1}^{T}X_t \tag{20}$$

where T is the total number of observations. Clearly $\bar{X}$ is unbiased. The variance of $\bar{X}, \sigma_{\bar{x}}^2$, is

$$\sigma_{\bar{x}}^2=E(\bar{X}-\mu_x)^2=E\left[\frac{1}{T}\sum_{t=1}^{T}(X_t-\mu_x)\right]^2$$

$$=\frac{1}{T^2}\sum_{i=1}^{T}\sum_{j=1}^{T}E(X_i-\mu_x)(X_j-\mu_x)$$

$$=\frac{1}{T^2}\sum_{i=1}^{T}\sum_{j=1}^{T}\sigma_{xx}(i-j).$$

$$=\frac{1}{T}\sigma_{xx}(0)+\frac{2}{T^2}\sum_{i=1}^{T-1}(T-i)\sigma_{xx}(i). \tag{21}$$

The estimator $\bar{X}$ is consistent, if and only if $\sigma_{\bar{x}}^2$ tends to zero as T goes to infinity. So whether $\bar{X}$ is consistent or not depends upon the structure of autocovariances of the series $\{X_t\}$.

We have,

Lemma 1. The estimator $\bar{X}$ is consistent, if for any $\varepsilon>0$, there exists an integer N_ε such that $|\sigma_{xx}(i)|\leq\varepsilon$ for all $i\geq N_\varepsilon$.

Proof. Let $\varepsilon=\frac{1}{k}$ and $T=kN_\varepsilon$ for $k>1$. Then,

$$\frac{2}{T^2}\sum_{i=1}^{T-1}(T-i)\sigma_{xx}(i)=\frac{2}{T^2}\sum_{i=1}^{N_\varepsilon}(T-i)\sigma_{xx}(i)+\frac{2}{T^2}\sum_{i=N_\varepsilon+1}^{T-1}(T-i)\sigma_{xx}(i)$$

$$\leq\frac{2\sigma_{xx}(0)}{T^2}\,\frac{(2T-N_\varepsilon-1)N_\varepsilon}{2}+\frac{2\varepsilon}{T^2}\,\frac{(T-N_\varepsilon)(T-N_\varepsilon-1)}{2}$$

$$=\sigma_{xx}(0)\left(\frac{2}{k}-\frac{1}{k_2}-\frac{1}{k^2N_\varepsilon}\right)+\frac{1}{k}\left(1-\frac{2}{k}+\frac{1}{k^2}-\frac{1}{kN_\varepsilon}+\frac{1}{k^2N_\varepsilon}\right)$$

Notice that N_ε is non-decreasing as k goes to infinity. Therefore,

$$\lim_{T\to\infty}\frac{2}{T^2}\sum_{i=1}^{T-1}(T-i)\sigma_{xx}(i)=0$$

and $\bar{X}$ is consistent. Q.E.D.

Lemma 1 is different from Theorem 8. 3. 1 in Anderson[1], which requires $\lim_{T\to\infty}\sum_{i=1}^{T}\sigma_{xx}(i)<\infty$. First, consider a series $\{X_t\}$ which has the autocovariances $\sigma_{xx}(i)=\frac{1}{i+1}$ for all $i\geq0$. In this case, the estimator $\bar{X}$ is a consistent estimator of μ_x by Lemma 1, since for any ε (say $\frac{1}{k}$), there exists an integer N_ε(say k) such that $|\sigma_{xx}(i)|<\frac{1}{k}$ for all $i\geq k$. However, Anderson's Theorem cannot establish the consistency, since $\lim_{T\to\infty}\sum_{i=1}^{T}\frac{1}{i+1}=\infty$. Next, let the series $\{X_t\}$ have the autocovariances such that $\sigma_{xx}(i)=(-1)^ic$ for all $i\geq1$ and for a constant $c>0$. Lemma 1 cannot establish the consistency of $\bar{X}$, since the $\sigma_{xx}(i)$ does not converge to zero as i goes to infinity. But the fact that $\sum_{i=1}^{T}\sigma_{xx}(i)$ is finite for all T makes $\bar{X}$ a consistent estimator of μ_x by Anderson's Theorem. So we have,

Theorem 1. The estimator $\bar{X}$ is consistent, if

1) $\sum_{i=1}^{T}\sigma_{xx}(i)<\infty$ for all T, or

2) for any $\varepsilon>0$. there exists an integer N_ε such that $|\sigma_{xx}(i)|\leq\varepsilon$ for all $i\geq N_\varepsilon$.

Even when the estimator $\bar{X}$ is consistent, we need very large number of observations in practice in order to obtain a useful estimate on μ_x. To see this descriptively, let the series $\{X_t\}$ be a first order Markov process with $\mu_x=0$ and $X_t=0.8X_{t-1}+\varepsilon_t$ where ε_t is a normal random variable with $\mu_\varepsilon=0$. Suppose X_0 happen to be 1. Then X_1 is a normal random variable with mean 0.8, and $P_r(X_1\geq 0)>P_r(X_1<0)$. This tendency will last until a subsequent X_t happens to realize as a negative quantity. When this tendency lasts long enough so that all the observations are made for these periods, then the computed $\bar{X}$ will certainly be a positive number no matter how many observations are made. Another example is that X_t is a moving average of one thousand observations of the white noise series, where the weights are all positive. Then the $\sigma_{xx}(i)$ is positive for all i from 1 to 1,000. In this case the estimate $\bar{X}$ from one thousand observations of X_t cannot provide a useful estimate on μ_x. Thus we need an extraordinarily large number of observations to obtain a useful estimate on μ_x, even when the $\bar{X}$ is consistent.

Since the estimator $\bar{X}$ is a weighted average of X_t, where the weights are all identical, one may raise a natural question: is there an optimal weighting scheme, other than $\frac{1}{T}$, to obtain a better estimate on μ_x? The answer is theoretically yes, but practically no. Let $X_{(T)}$ be the $T\times 1$ column vector of $(X_1, X_2, \ldots, X_T)$, U be the $T\times 1$ column vector of 1's, and α be a $T\times 1$ column vector of constants such that $\alpha' U=1$. Then

$$M_x=X_{(T)}'\alpha \tag{22}$$

is an unbiased estimator on μ_x. In following Lemma 2, we show that the estimator $\bar{X}$ is the best among alternatives M_x's when X_t's are pairwise uncorrelated, but not in general. Let Ω_{xx} be the $T\times T$ autocovariance matrix of $\{X_t\}$, which is non-singular.

Lemma 2. The estimator M_x with $\alpha=\frac{\Omega_{xx}^{-1}U}{U'\Omega_{xx}^{-1}U}$ is the minimum variance unbiased estimator on μ_x among the alternative estimators in the class of (22) The minimum variance of M_x is $\frac{1}{U'\Omega_{xx}^{-1}U}$.

Proof. Let $\sigma_{M_x}^2$ be the variance of M_x. Then

$$\sigma_{M_x}^2 = E(\alpha' X_{(T)} - \mu_x)\ (X_{(T)}'\alpha - \mu_x)$$
$$= \alpha' E(X_{(T)} - \mu_x U)\ (X_{(T)} - \mu_x U)'\alpha$$
$$= \alpha'\Omega_{xx}\alpha$$

Choose α to minimize $\sigma_{M_x}^2$ subject to $\alpha' U = 1$. Then the Lagrangean is

$$L = \alpha'\Omega_{xx}\alpha + \lambda(1 - \alpha' U),$$

and the first order condition gives

$$\frac{\partial L}{\partial \alpha} = 2\Omega_{xx}\alpha^* - \lambda^* U = 0$$
$$\frac{\partial L}{\partial \lambda} = 1 - \alpha^{*\prime} U = 0. \qquad (23)$$

It follows that

$$\lambda^* = \frac{2}{U'\Omega_{xx}^{-1}U} \text{ and } \alpha^* = \frac{\Omega_{xx}^{-1}U}{U'\Omega_{xx}^{-1}U} \text{ and } \sigma_{M_x}^{2*} = \alpha^{*\prime}\Omega_{xx}\alpha^* = \frac{1}{U'\Omega_{xx}^{-1}U}. \quad \text{Q.E.D.}$$

Now it is clear that the estimator $\bar{X}$ is the minimum variance unbiased estimator if and only if $\Omega_{xx} = \sigma^2 I$, namely the series $\{X_t\}$ is pairwise uncorrelated. Thus the estimator $\bar{X}$ is not efficient when the series $\{X_t\}$ is autocorrelated. Table 1 summarizes the results of simulated works to demonstrate the loss of efficiency for the estimator $\bar{X}$. The smaller the autocorrelation coefficient is (i.e., the closer to $\sigma^2 I$ the Ω_{xx} becomes), the better the estimator $\bar{X}$ becomes, which verifies the implication of Lemma 2. This leads to the problem of the GLS estimation of Aitken. So in the next paragraph we will restrict the problem to a simple case such that we can follow the Durbin's procedure.

Since the optimal weighting scheme in Lemma 2 requires *a priori* knowledge of the auto-covariance matrix Ω_{xx}, it is not possible in practice to design an efficient estimator on the mean μ_x. On the other hand, the estimator $\bar{X}$ is not useful at least for small sample size, since almost any empirical time series is autocorrelated. Therefore, it may be better to assume that the series is generated by a first order Markov process with unknown serial correlation coefficient. For the series $\{X_t\}$, we may assume

$$X_t=(1-\rho)\mu_x+\rho X_{t-1}+\varepsilon_t, \tag{24}$$

where ε_t is a white noise and not correlated with X_{t-1}. Then,

$$\Omega_{xx}=\sigma_x^2 P_{xx} \tag{25}$$

where

$$P_{xx}=\begin{pmatrix} 1 & \rho & \rho^2\cdots\cdots\rho^{T-1} \\ \rho & 1 & \rho\cdots\cdots\rho^{T-2} \\ \rho^2 & \rho & 1\cdots\cdots\rho^{T-3} \\ \vdots & \vdots & \vdots \qquad\quad \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3}\cdots\cdots 1 \end{pmatrix}. \tag{26}$$

It can be shown that

$$P_{xx}^{-1}=\frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho & 0\cdots\cdots 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho\cdots\cdots 0 & 0 & 0 \\ 0 & -\rho & 1+\rho^2\cdots\cdots 0 & 0 & 0 \\ \vdots & \vdots & \vdots \qquad \vdots & \vdots & \vdots \\ 0 & 0 & 0\cdots\cdots -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0\cdots\cdots 0 & -\rho & 1 \end{pmatrix}. \tag{27}$$

Then

$$\alpha=\frac{\Omega_{xx}^{-1}U}{U'\Omega_{xx}^{-1}U}=\frac{(1-\rho^2)P_{xx}^{-1}U}{(1-\rho^2)U'P_{xx}^{-1}U}. \tag{28}$$

Now

$$(1-\rho^2)P_{xx}^{-1}U=(1-\rho)\begin{pmatrix} 1 \\ 1-\rho \\ \vdots \\ 1-\rho \\ 1 \end{pmatrix},$$

and $$(1-\rho^2)\ U'P_{xx}^{-1}U=(1-\rho)\ (T-(T-2)\ \rho). \tag{29}$$

Therefore we have

$$\begin{aligned} a_i&=\frac{1}{T-(T-2)\rho} \text{ if } i=1 \text{ or } T, \\ &=\frac{1-\rho}{T-(T-2)\rho} \text{ if } 2\leq i\leq T-1. \end{aligned} \tag{30}$$

Table 1. The sample means and variances of the first order Markov processes for each value of the serial correlation coefficient ρ.

ρ	μ_x	σ_x^2	Group A		Group B	
			$\bar{X}$	S_x^2	$\bar{X}$	S_x^2
0.0	0	1	0.0090	0.9873	−0.2013	1.0753
0.2	0	1	0.0150	0.9818	−0.2416	1.0853
0.5	0	1	0.0256	0.9444	−0.3466	1.0073
0.9	0	1	0.8200	0.7381	−0.7483	0.3059

The least squares estimate on ρ in (24) may be used to compute i's in (30). This procedure also may be iterated.

2. *The Autocovariances.* The usual estimator on the autocovariance for lag τ is $S_{xx}(\tau)$ such that

$$S_{xx}(\tau)=\frac{1}{T}\sum_{t=1}^{T-\tau}(X_t-\bar{X})\ (X_{t+\tau}-\bar{X}). \tag{31}$$

The mathematical expectation of $S_{xx}(\tau)$ is

$$\begin{aligned}ES_{xx}(0)&=E\left[\sum_{t=1}^{T}(X_t-\bar{X})^2\right]\\&=\frac{1}{T}E\left[\sum_{t=1}^{T}(X_t-\mu)^2\right]-\frac{2}{T}E\left[\sum_{t=1}^{T}(X_t-\mu_x)(\bar{X}-\mu_x)\right]+E(\bar{X}-\mu_x)^2\\&=\sigma_x^2-\sigma_{\bar{X}}^2,\end{aligned} \tag{32}$$

$$\begin{aligned}ES_{xx}(\tau)&=\frac{1}{T}E\left[\sum_{t=1}^{T-\tau}(X_t-\bar{X})(X_{t+\tau}-\bar{X})\right]\\&=\frac{1}{T}E\left[\sum_{t=1}^{T-\tau}(X_t-\mu_x)(X_{t+\tau}-\mu_x)\right]+\frac{T-\tau}{T}E(\bar{X}-\mu_x)^2\\&\quad-\frac{1}{T}E\left[\sum_{t=1}^{T-\tau}(X_t-\mu_x)(\bar{X}-\mu_x)\right]-\frac{1}{T}E\left[\sum_{t=1}^{T-\tau}(X_{t+\tau}-\mu_x)(\bar{X}-\mu_x)\right]\\&=\frac{T-\tau}{T}\sigma_{xx}(\tau)+\frac{T-\tau}{T}\sigma_{\bar{X}}^2-\frac{1}{T}\left[T\sigma_{\bar{X}}^2-\frac{1}{T}\sum_{t=T-\tau+1}^{T}\sum_{i=1}^{T}\sigma_{xx}(i-t)\right]\\&\quad-\frac{1}{T}\left[T\sigma_{\bar{X}}^2-\frac{1}{T}\sum_{t=1}^{\tau}\sum_{i=1}^{T}\sigma_{xx}(i-t)\right]\\&=(1-\frac{\tau}{T})\sigma_{xx}(\tau)-(1+\frac{\tau}{T})\sigma_{\bar{X}}^2+\frac{2}{T^2}\sum_{t=1}^{\tau}\sum_{i=1}^{T}\sigma_{xx}(i-t).\end{aligned} \tag{33}$$

Ignoring the last term in (33), the estimator $S_{xx}(\tau)$ is biased downward for all τ, and the size of bias increases as τ gets greater. Notice that the estimator $S_{xx}(\tau)$ is asymptotically unbiased only if $\sigma_{\bar{X}}^2$ tends to zero as T goes to infinity, namely the estimator $\bar{X}$ is consistent. Since the magnitude of $\sigma_{\bar{X}}^2$ does not decrease rapidly in general as T increases, we should expect a significant bias for small samples. Then the difference between biases for different lags becomes a serious problem, which may create serious distortions to the overall shape of the autocovariance structure. Frequently one observes empirical correlograms of Markov processes with the positive serial correlations, which exhibit the tendency of monotone decreasing below zero as the lag τ increases. This may have been caused by the increasing downward bias of $S_{xx}(\tau)$ along the lag τ.

If we use $\hat{S}_{xx}(\tau)$ instead of $S_{xx}(\tau)$, where

$$\hat{S}_{xx}(\tau)=\frac{1}{T-\tau}\sum_{t=1}^{T-\tau}X_tX_{t+\tau}-\bar{X}^2, \tag{34}$$

then

$$E\hat{S}_{xx}(\tau)=\frac{1}{T-\tau}\sum_{t=1}^{T-\tau}E(X_t-\mu_x)(X_{t+\tau}-\mu_x)-E(\bar{X}-\mu_x)^2=\sigma_{xx}(\tau)-\sigma_{\bar{x}}^2 \tag{35}$$

Notice that $\hat{S}_{xx}(\tau)$ exhibits uniform downard biases, i.e., $-\sigma_{\bar{x}}^2$, across all lags and therefore the bias of $\hat{S}_{xx}(\tau)$ is smaller than that of $\hat{S}_{xx}(\tau)$.

If we use $Q_{xx}(\tau)$, where

$$Q_{xx}(\tau)=\frac{1}{T-\tau}\sum_{t=1}^{T-\tau}X_tX_{t+\tau}-M_x^2, \tag{36}$$

then

$$EQ_{xx}(\tau)=\sigma_{xx}(\tau)-\sigma_{Mx}^{\cdot}. \tag{37}$$

In general the estimator $Q_{xx}(\tau)$ will incur the smaller bias than the $\hat{S}_{xx}(\tau)$, since σ_{Mx}^2 is smaller than $\sigma_{\bar{x}}^2$.

It doesn't follow, however, that the estimator $Q_{xx}(\tau)$ will always prove better estimates in practice, since the above analysis does not include the covariances of the estimators. It is shown in Bartlett [2] that the estimates $S_{xx}(\tau)$ and $S_{xx}(\tau+i)$ (or $\hat{S}_{xx}(\tau)$ and $\hat{S}_{xx}(\tau+i)$, $Q_{xx}(\tau)$ and $Q_{xx}(\tau+i)$) are correlated, and some authors (e.g., Box and Newbold [4]) attribute the smoothness of empirical correlograms to this fact. The fact that the estimates are correlated with each other is often indicated to warn the approaches to study the overall shape of correlogram, which is our main interest here. But the estimators $S_{xx}(\tau)$ and $S_{xx}(\tau+i)$ are based on the identical sample $(X_1, X_2, \ldots, X_T)$, and therefore they are naturally correlated, just as $\bar{X}$ is correlated with $S_{xx}(0)$ for any case. The smoothness of correlogram may be partly due to this problematic correlation, but also is due to the smoothness of the true correlogram. Recalling that the correlation coefficient measures a degree of linear co-deviation away from the mean, it is clear that an abrupt discontinuity of the true correlogram can easily destroy the smoothness of empirical correlogram. In fact, one can obtain a smoother empirical correlogram by simply increasing the sample size when the true correlogram is smooth. Notice that the covariance between $S_{xx}(\tau)$ and $S_{xx}(\tau+i)$ decreases as the sample size increases. This may be understood that the smoothness of empirical correlogram is mainly due to the smoothness of the true correlogram rather than the problematic correlation between the estimates.

Another point can be raised on the shape of the empirical correlogram. The estimated autocorrelation coefficient $r_{xx}(\tau)$ is

$$r_{xx}(\tau)=\frac{S_{xx}(\tau)}{S_{xx}(0)} \tag{38}$$

When the value of $S_{xx}(0)$ happens to be unduly smaller than the true variance σ_x^2, which is usually the case for a highly positively autocorrelated series, then the empirical autocorrelation coefficient $r_{xx}(\tau)$ will be unduly amplified (see Table 1). But it will still preserve the relative overall shape of correlogram, if the $S_{xx}(\tau)$'s for $\tau\geq 1$ are relatively good estimates. All these problems are perhaps due to the large bias of the usual estimator $S_{xx}(\tau)$, which is caused by the poor efficiency of the estimator $\bar{X}$. This is why we thought the estimator $Q_{xx}(\tau)$ behaves better than $S_{xx}(\tau)$. But the results of our simulated works are very discouraging in that we could not find any improvement in either $Q_{xx}(\tau)$ or M_x relative to $S_{xx}(\tau)$ or $\bar{X}$, even when the series X_t is a Markov process. We found that least squares estimates on ρ were so poor that the iterative method of (30) was not very useful.

3. *The Cross-Covariances.* The sampling properties of the cross-covariances can be shown in the similar way as in those of autocovariances (V. 2). Bartlett [2] has shown that for the linear processes the empirical cross-correlogram is best reliable, if each series is not autocorrelated. Box and Newbold [4] provided an example, which demonstrates relatively high cross-correlations between two independent series, if each series is highly positively autocorrelated. This finding may be explained on two aspects. First, depending upon the initial realization, each autocorrelated series (say $\rho=0.9$) will exhibit monotone increasing or decreasing trend sufficiently long, and consequently, the cross correlogram indicates significant covariations. Second, if each series is highly positively autocorrelated, then the estimated variance is so small that the empirical cross-correlogram is unduly

inflated, misleading one to suspect a strong linear relationship for the two series which are in fact independent. The Fig. 17 through Fig. 40 are the empirical cross correlograms between various independent pairs of autocorrelated series with $\rho=0.6$ or $\rho=0.99$, for the sample sizes $N=50$ and 100. We again found very high cross-correlations in these simulations, but it is interesting to notice that the locations of high cross-correlations are random and not systematic. Even for the same pair of series, the cross-correlograms look drastically different when we change the sample size. Notice that this is not the case when we simulated Fig. 1 through Fig. 9 where each pair of series are generated by each causal structure. So it may be true that if we observe the same location of maximum for each empirical cross-correlogram obtained from different sample size, then it indicates a certain underlying causal structure, otherwise not. But this idea is a pure conjecture at this moment, and perhaps it is worth paying attention to this aspect.

VI. Concluding Remarks

So far we tried to provide a theorization to the Hooker-Campbellian idea, which proposed the use of the cross-correlogram in the study of the causal structure between two timeseries. We were successful to demonstrate that the structure of the true cross correlogram is consistent with the Hooker-Campbellian claim. But this study can by no means be complete, because there is no known way to obtain a good estimate of the cross-correlogram. In spite of the poor statistical properties of the usual empirical cross-correlograms, however, the structure of the empirical cross-correlogram was consistent with the structure of the true correlogram for every simulation we performed, as far as the causal study is concerned. On the other hand, the shape of the empirical cross-correlogram is found to be quite irregular (specially the location of the maximum) for different number of observations, when the two series are independent.

We consider this aspect as a clue to solve the difficulties related to the estimation, and further research will be performed along with this line.

Figures

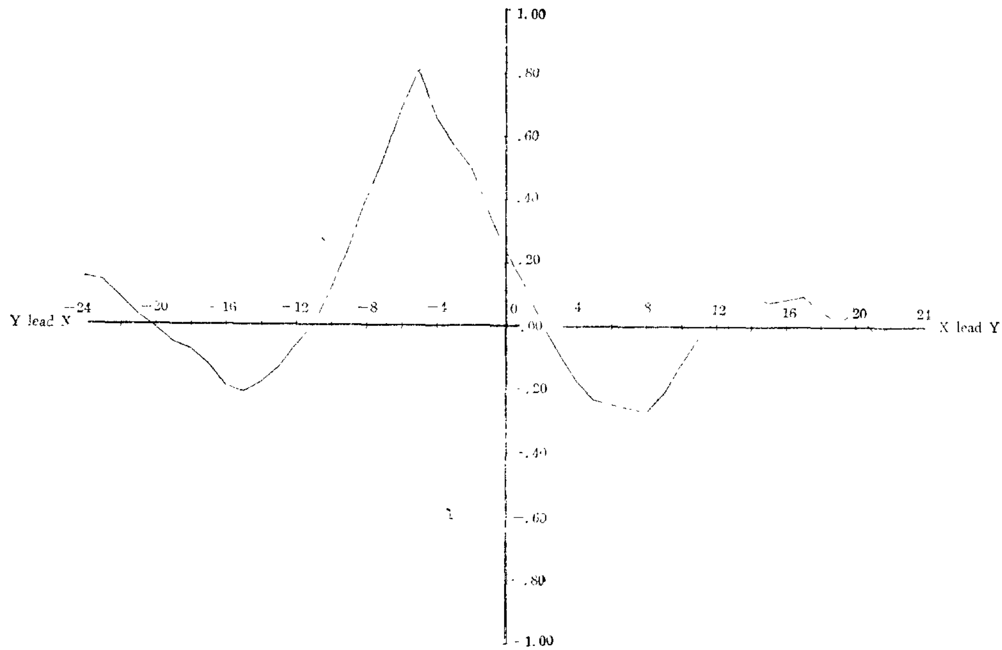


Fig.1. $X_t = -0.7Y_{t-5} + \varepsilon_t$, $\rho_{yy} = 0.8$

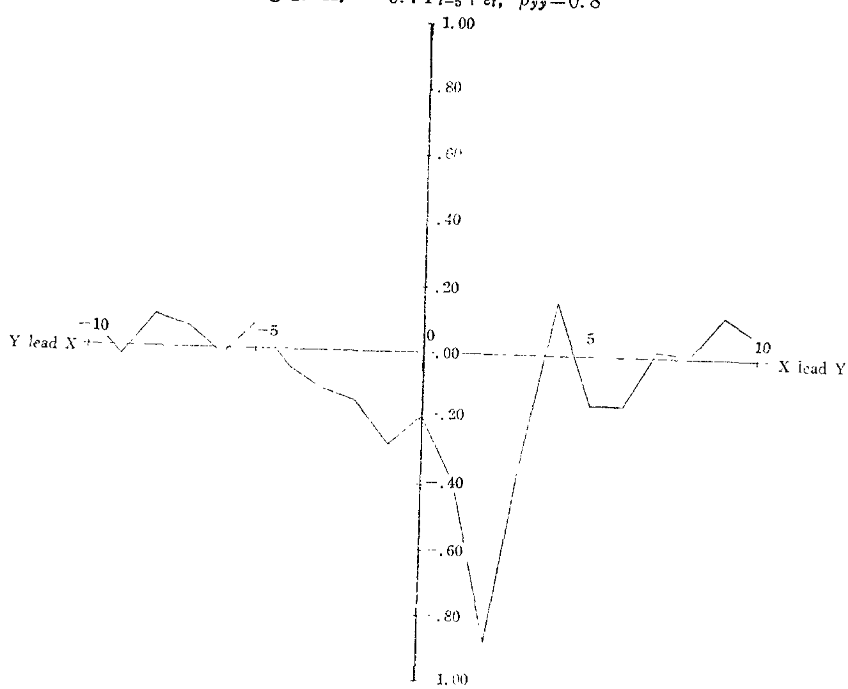


Fig.2. $Y_t = -2.5X_{t-2} + \varepsilon_t$, $\rho_{xx} = 0.8$, $\rho_{\varepsilon\varepsilon} = 0.7$

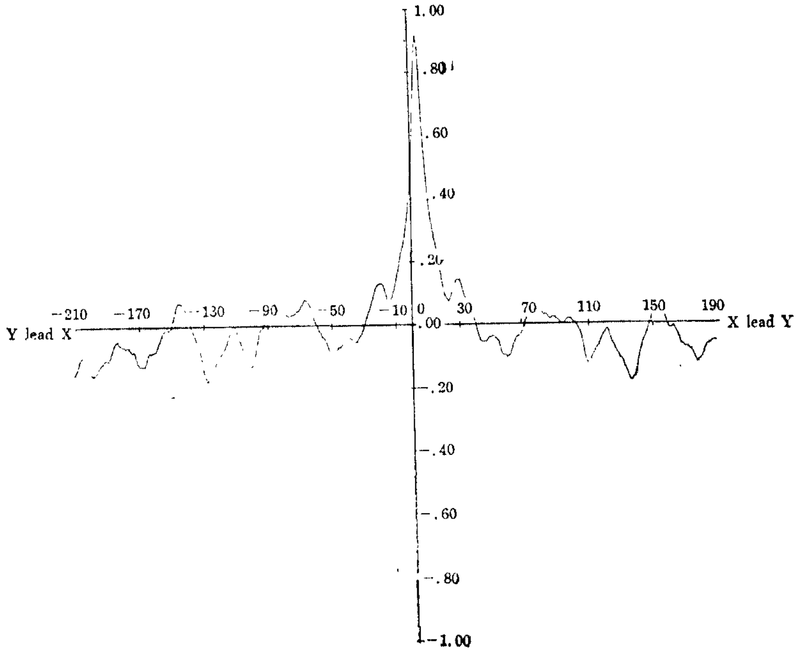


Fig.3. $Y_t = 1.7X_{t-2} + 1.2X_{t-3} + 1.0X_{t-4} + 0.7X_{t-5} + 0.2X_{t-6} + \varepsilon_t$, $\rho_{xx} = 0.8$

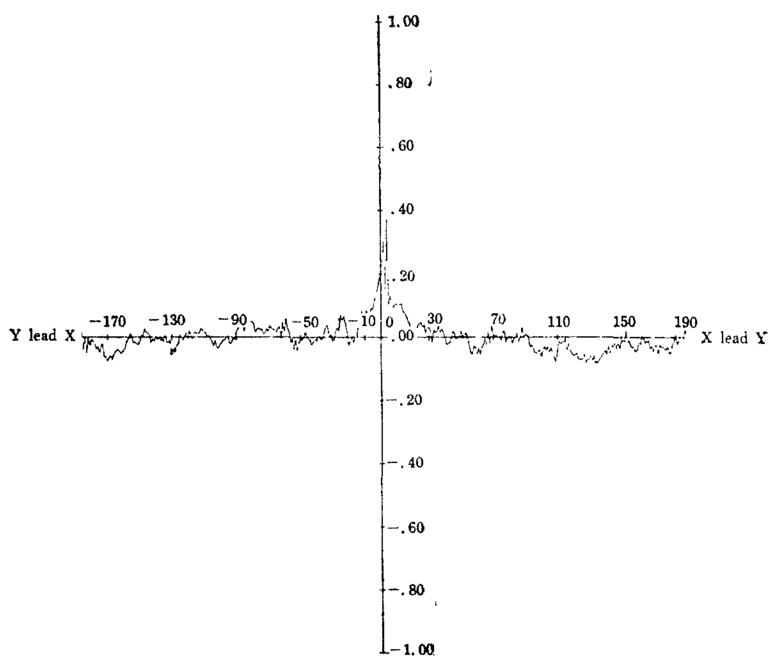


Fig.4. $Y_t = 0.7X_{t-2} - 1.2X_{t-3} + 1.7X_{t-4} - 1.0X_{t-5} + 0.2X_{t-6} + \varepsilon_t$, $\rho_{xx} = 0.8$

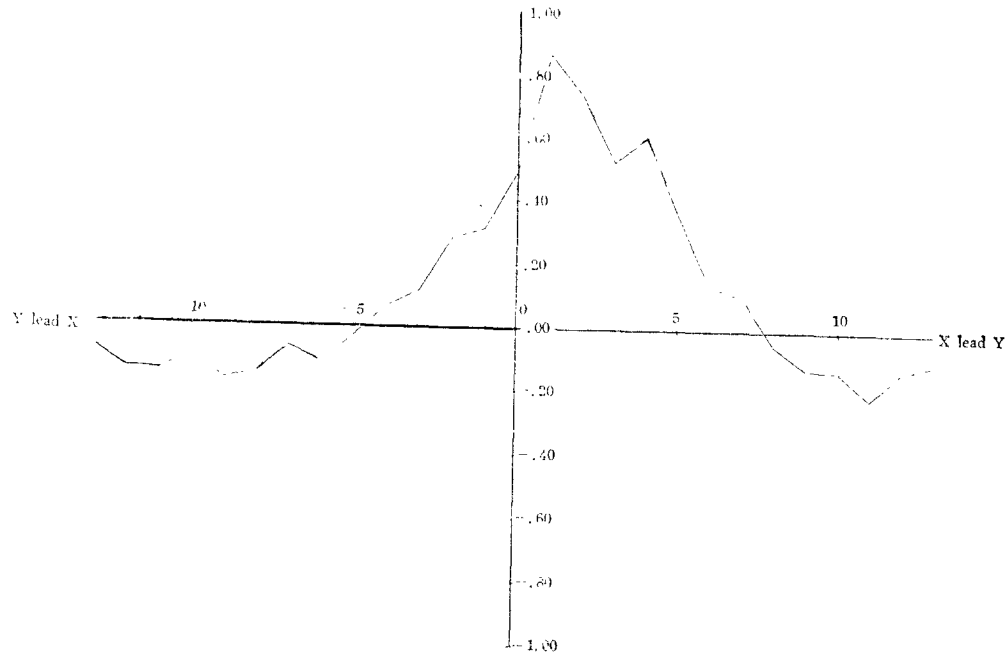


Fig.5. $Y_t=1.7X_{t-1}+1.1X_{t-2}+\varepsilon_t,\ \ \rho_{xx}=0.8,\ \ \rho_{\varepsilon\varepsilon}=0.7$

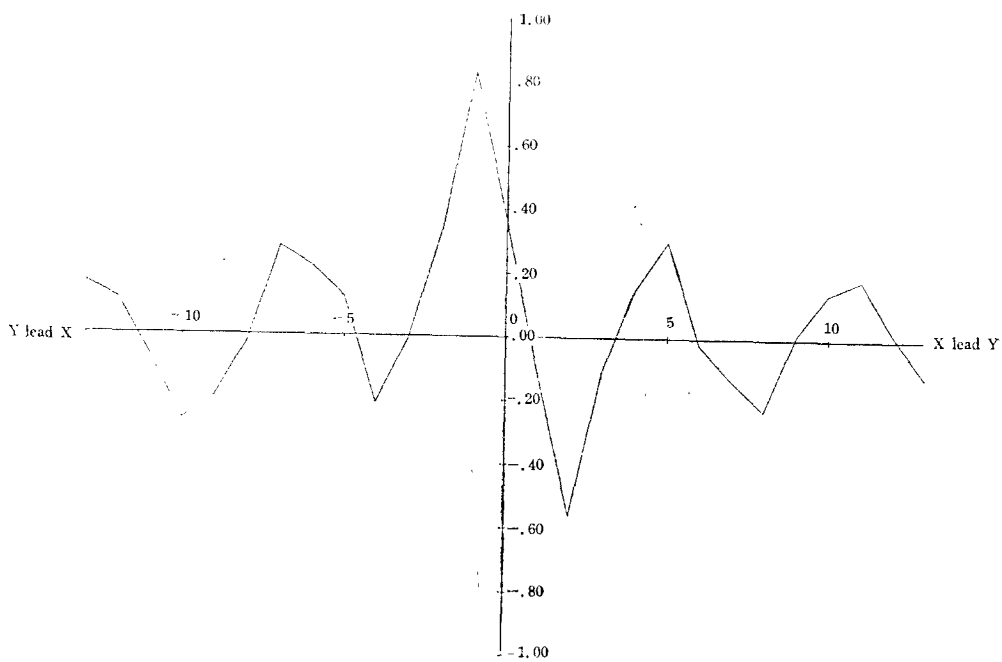


Fig.6. $X_t=0.8Y_{t-1}+\varepsilon_t,\ \ Y_t=-0.7X_{t-2}+\eta_t,\ \ \rho_{\varepsilon\varepsilon}=0.7,\ \ \rho_{\eta\eta}=0.8$

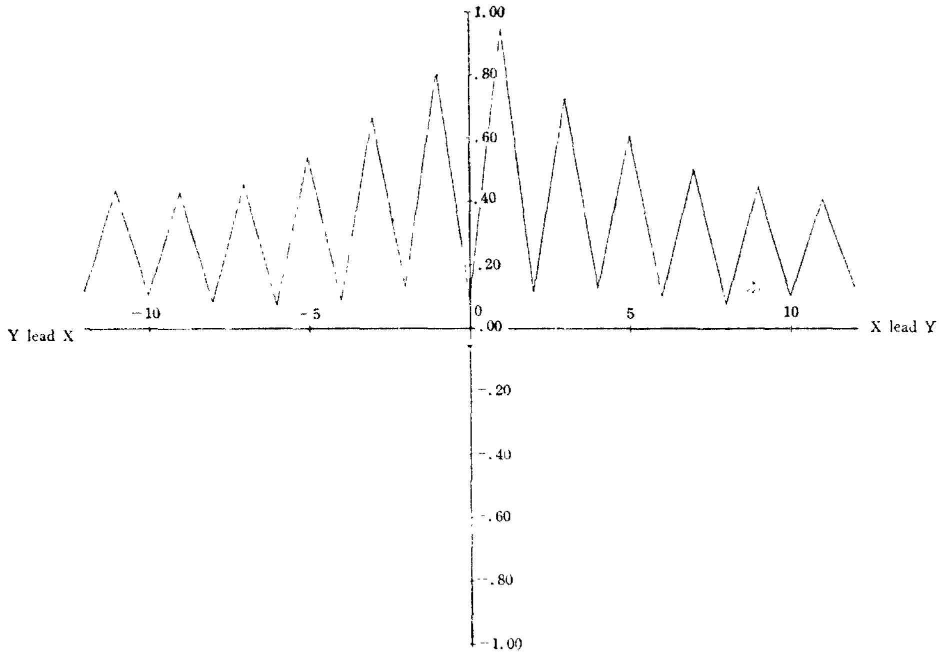


Fig.7. $X_t=0.3Y_{t-1}+\varepsilon_t,\ \ Y_t=2.7X_{t-1}+\eta_t$

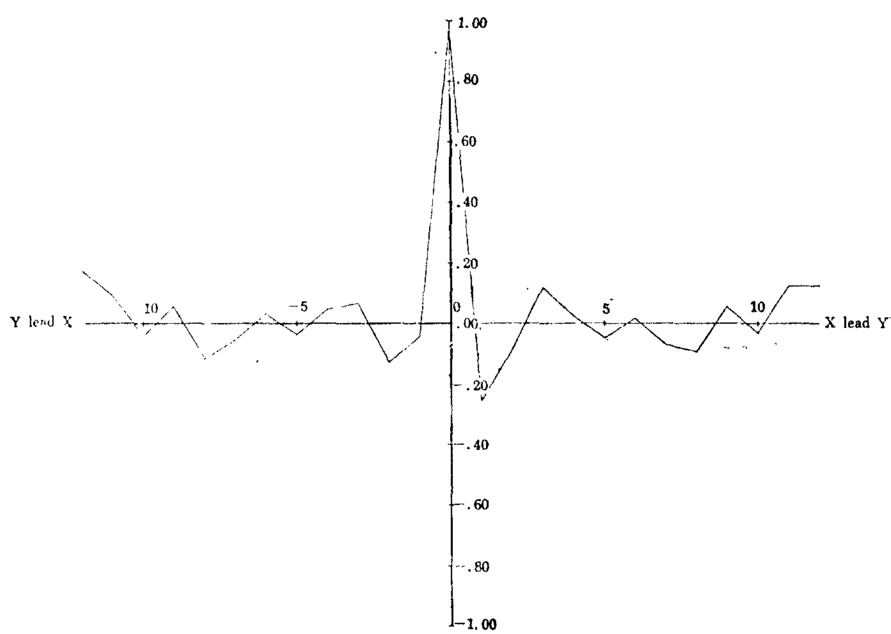


Fig.8. $X_t=1.7Y_t+\varepsilon_t,\ \ Y_t=2.1\varepsilon_{t-1}+\eta_t$

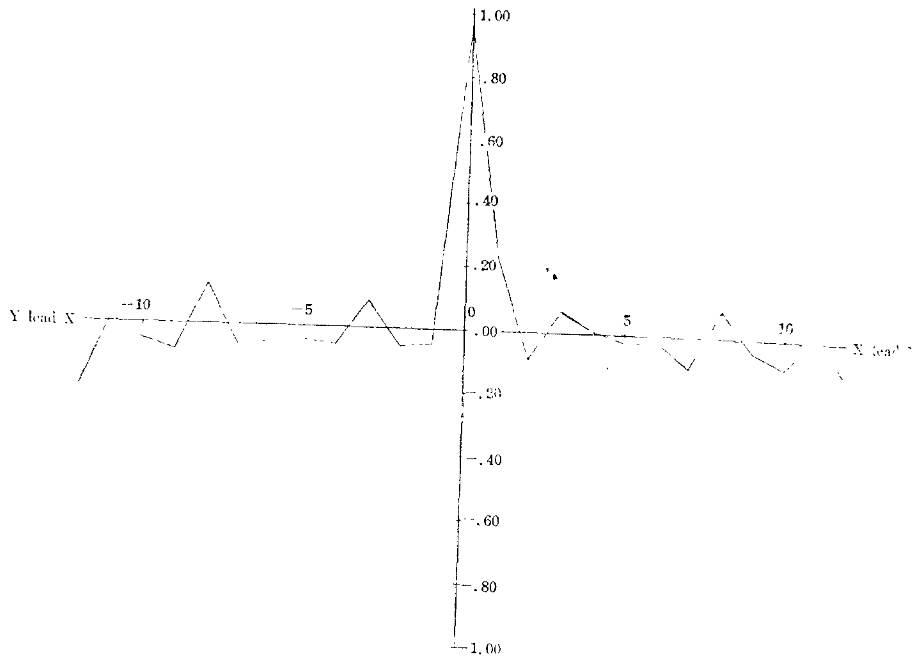


Fig.9. $X_t = 1.7Y_t + \varepsilon_t$, $Y_t = -2.1\varepsilon_{t-1} + \eta_t$.

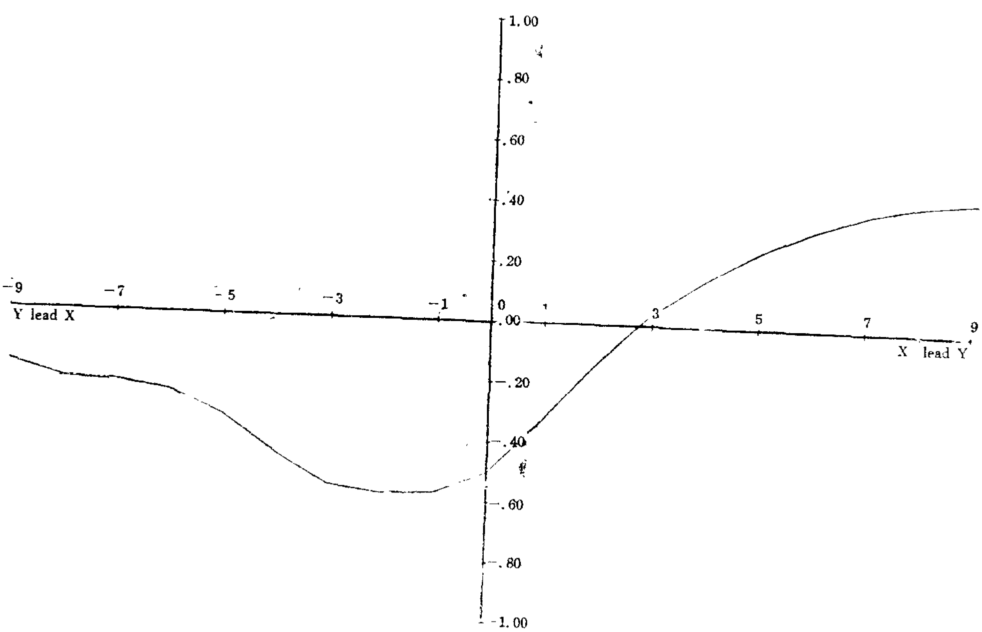


Fig.10. Male, X=Total Admissions to Mental Hospital, Y=Employment

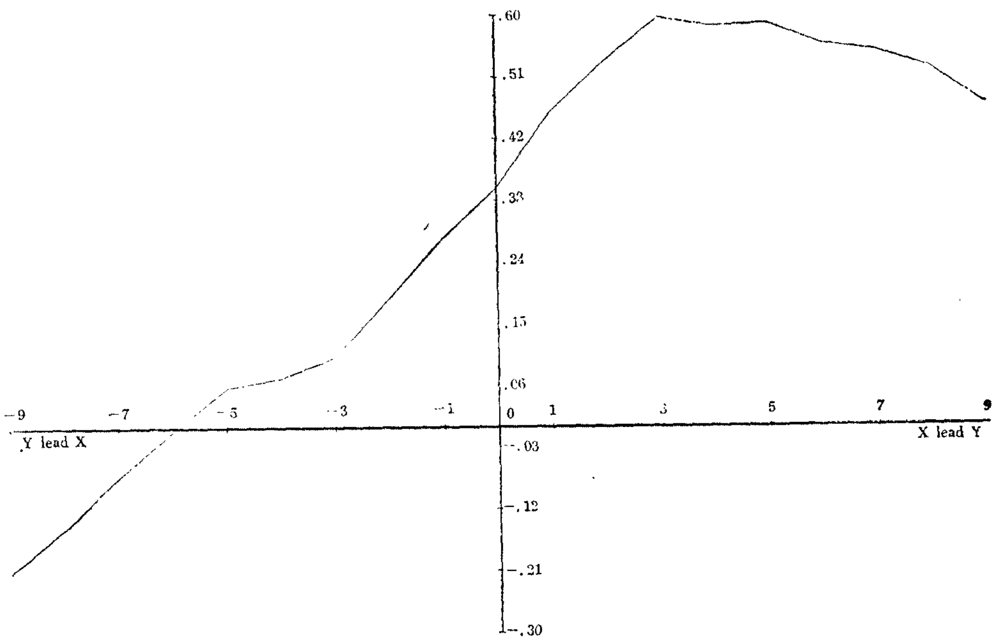


Fig.11. Female, X=Total Admissions to Mental Hospital, Y=Employment

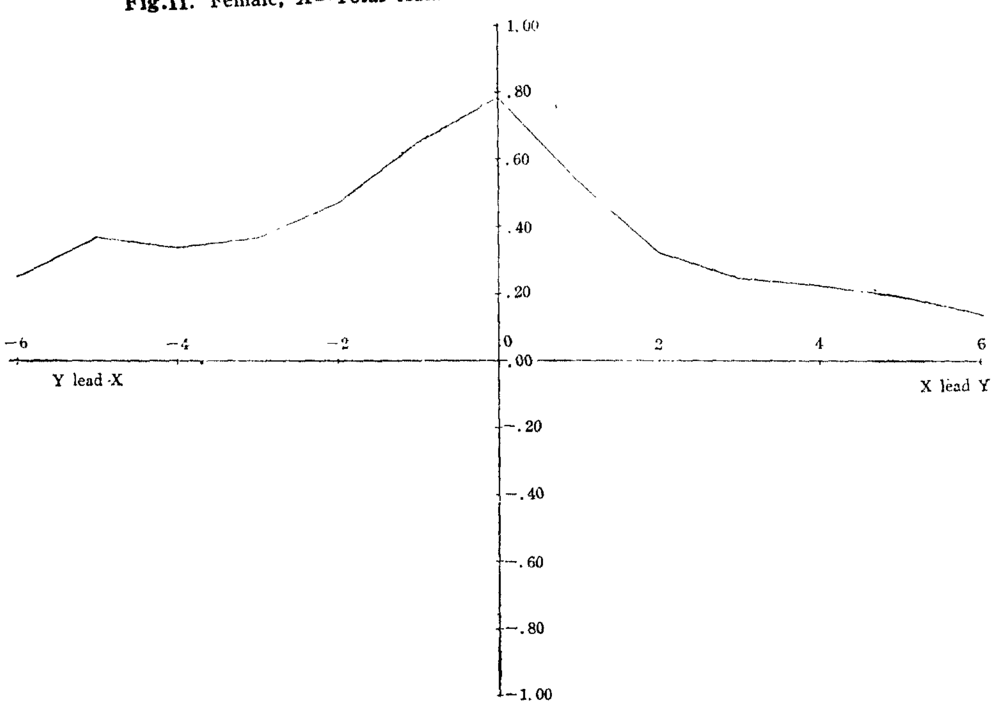


Fig. 12. Male, X=Cirrhosis, Y=Alcohol Consumption (1 year lagged)

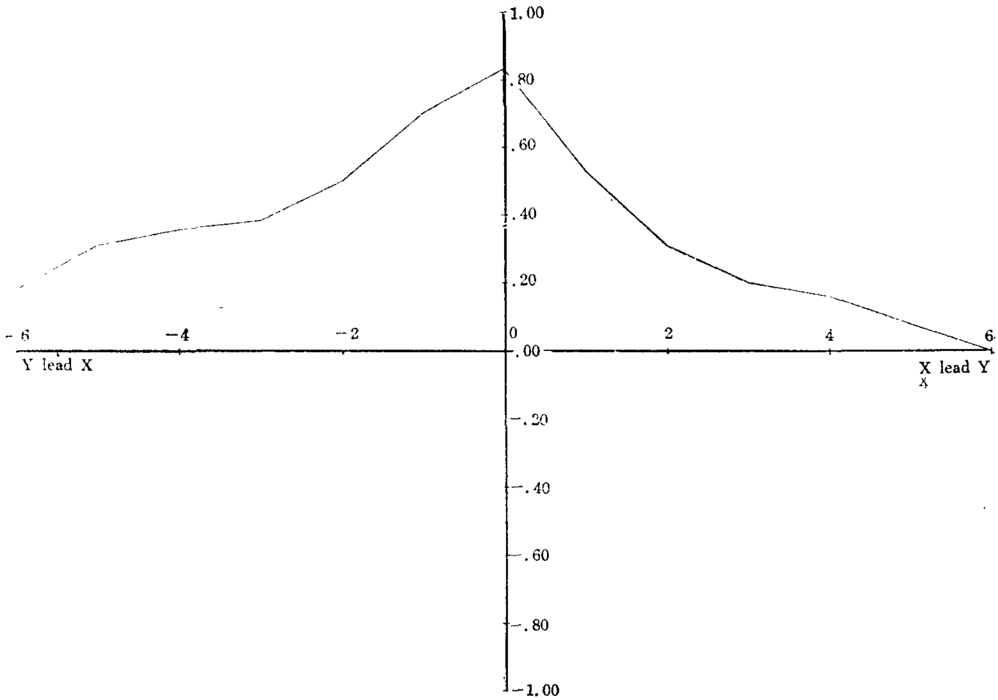


Fig.13. Female, X=Cirrhosis, Y=Alcohol Consumption (1 year lagged)

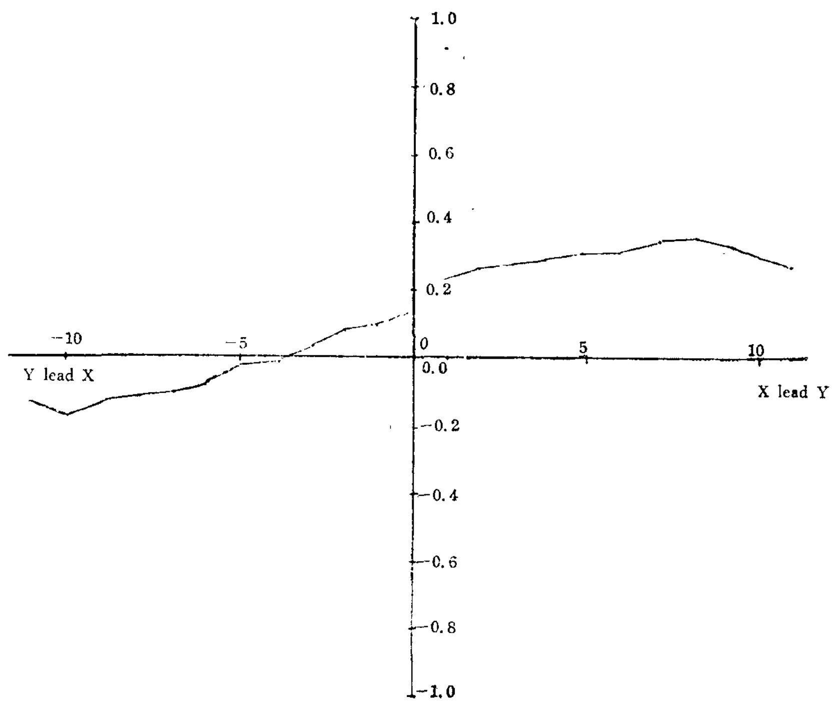


Fig.13-1. X=Annual Rate of Murders, Y=Probability of Execution (1933~73)

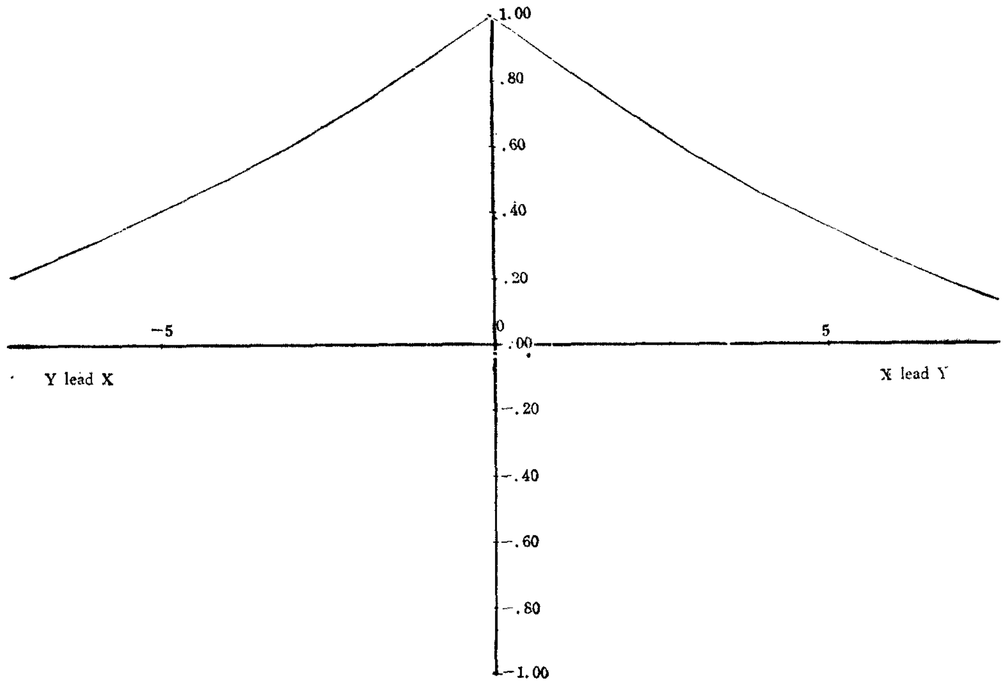


Fig.14. Y=GNP, X=Money Supply excluding Time Deposits (M_1), Annual (1947~74)

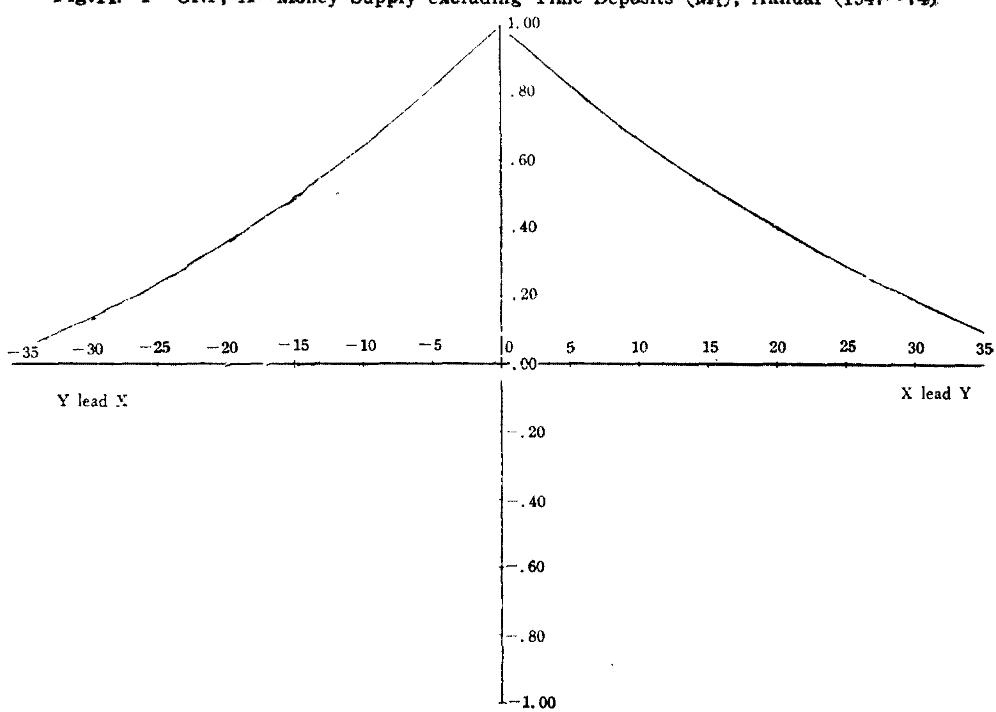


Fig.15. X=GNP, Y=Money Supply excluding Time Deposits (M_1), Quarterly (1947~74)

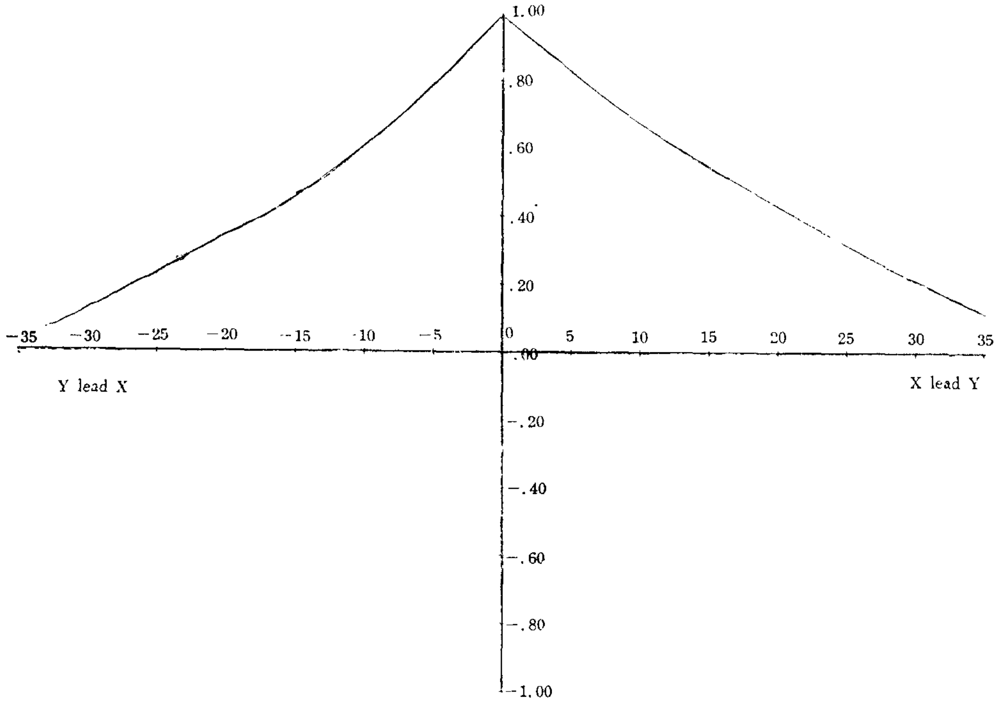


Fig.16. X=GNP, Y=Money Supply including Time Deposits (M_2), Quarterly (1947～74)

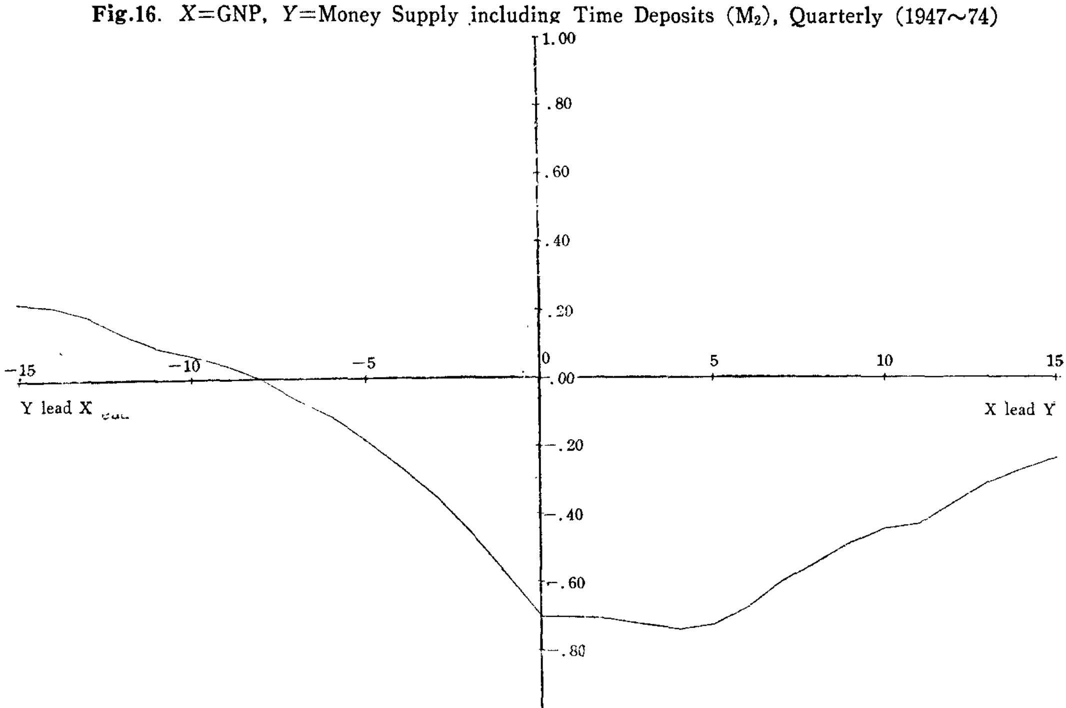


Fig.17. Pair A, ρ=0.99, N=50

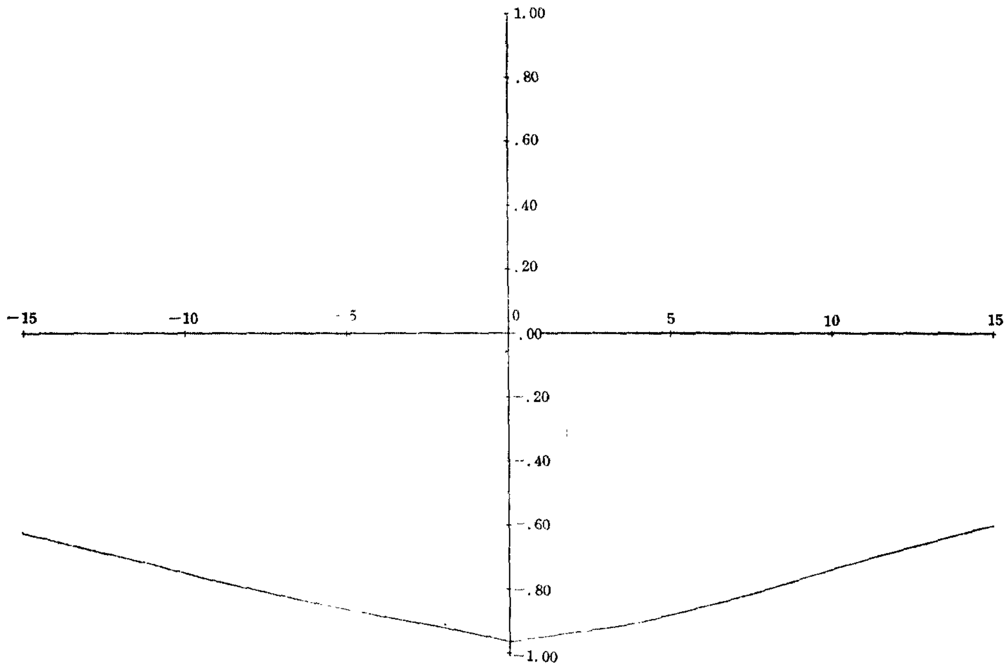


Fig.18. Pair A, ρ=0.99, N=100

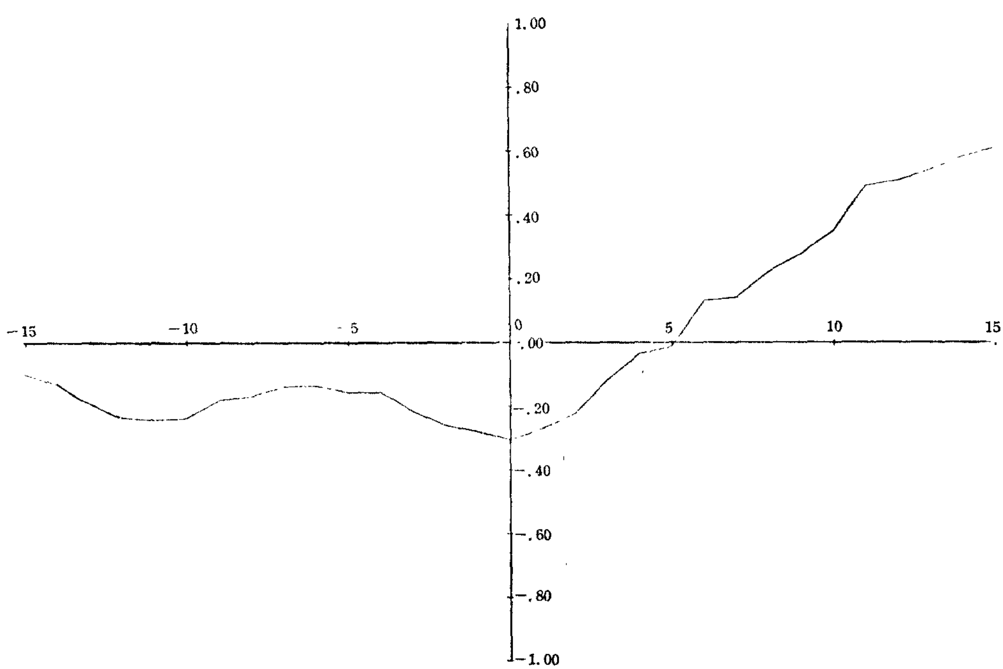


Fig.19. Pair B, ρ=0.99, N=50

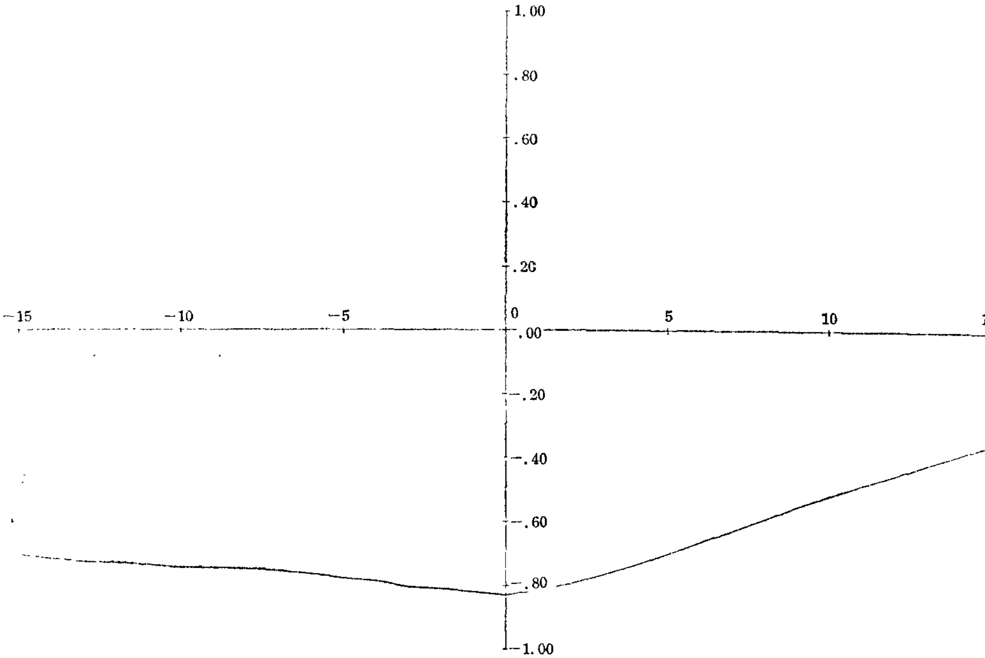


Fig.20. Pair B, $\rho=0.99$, $N=100$

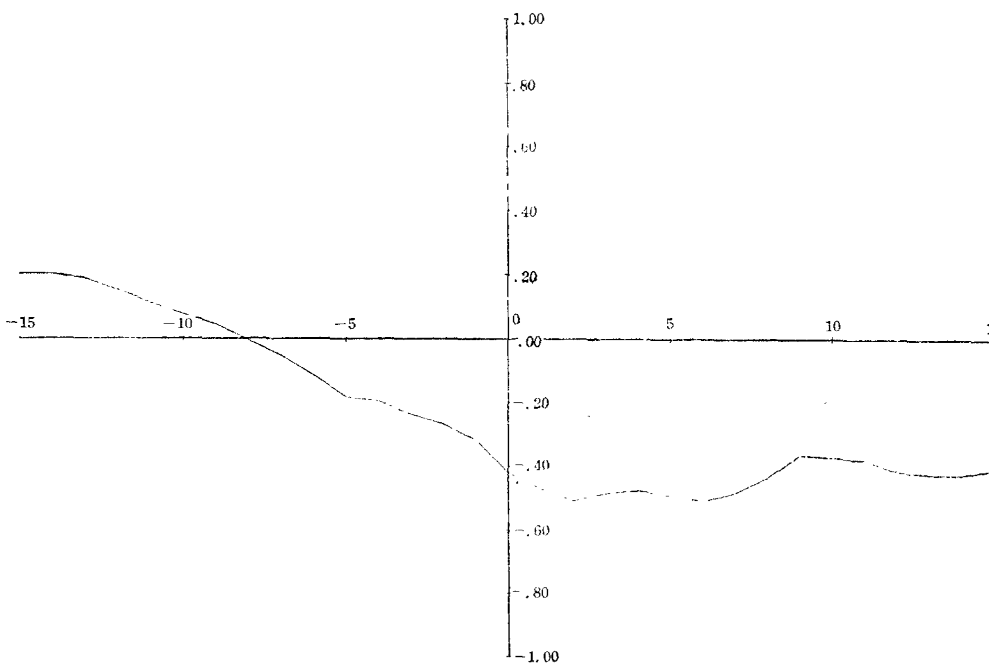


Fig.21. Pair C, $\rho=0.99$, $N=50$

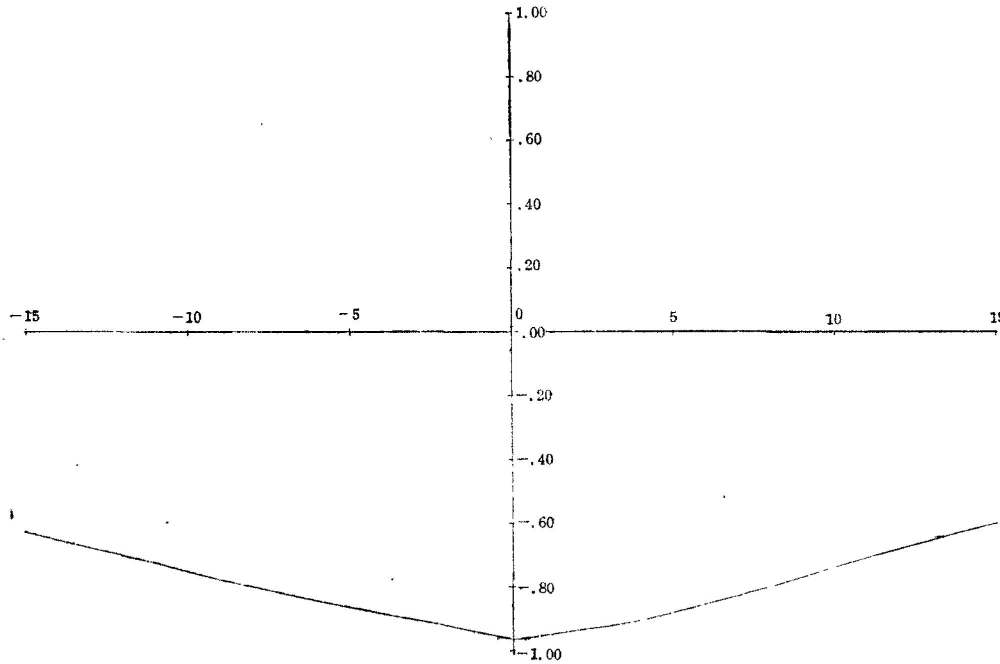


Fig.22. Pair C, $\rho=0.99$, $N=100$

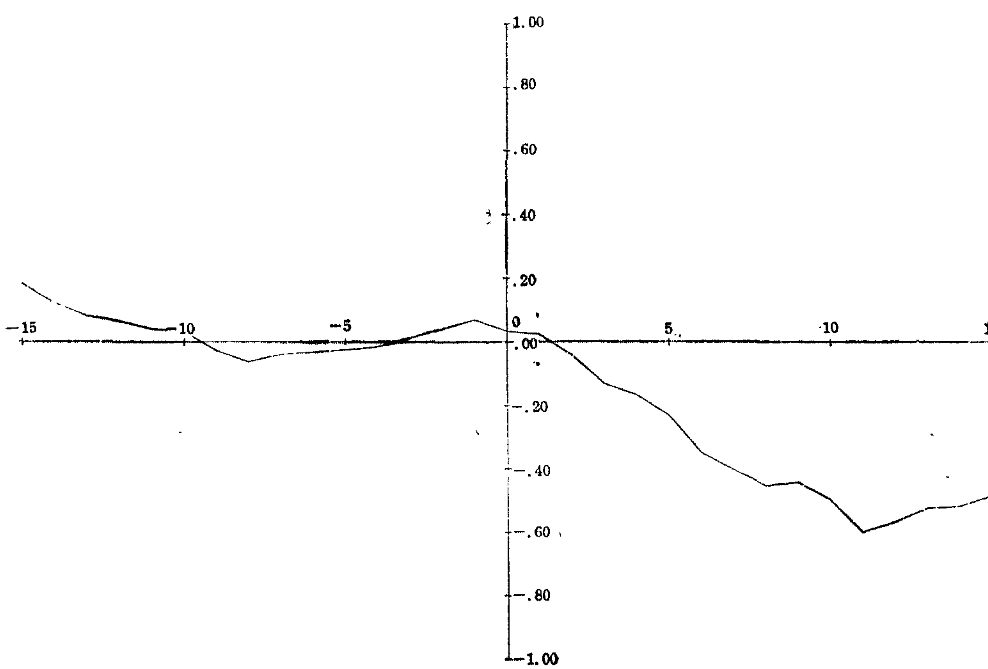


Fig.23. Pair D, $\rho=0.99$, $N=50$

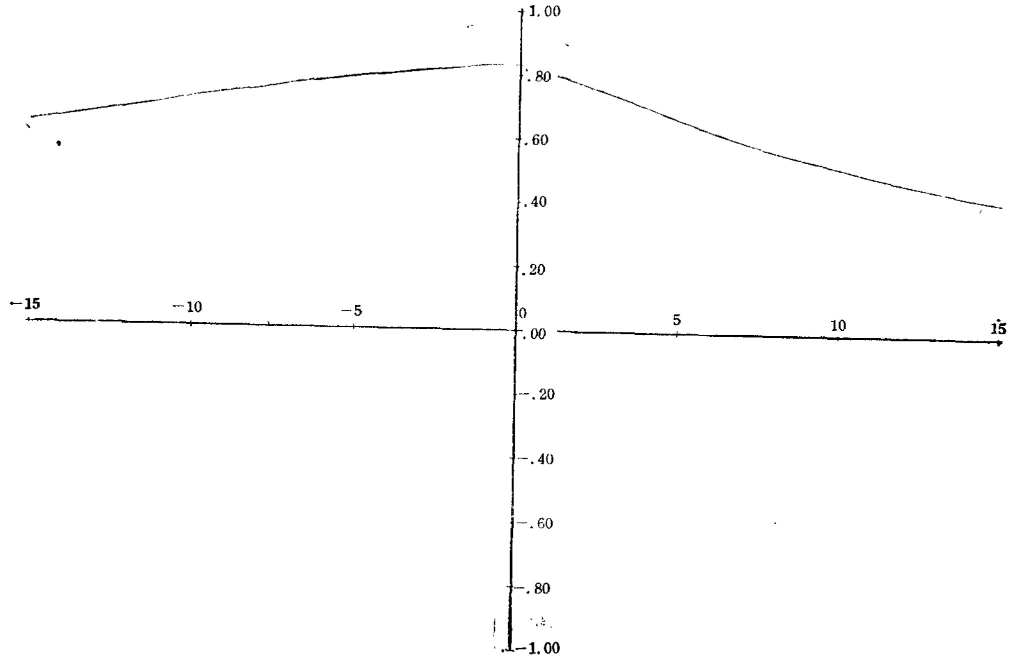


Fig.24. Pair D, ρ=0.99, N=100

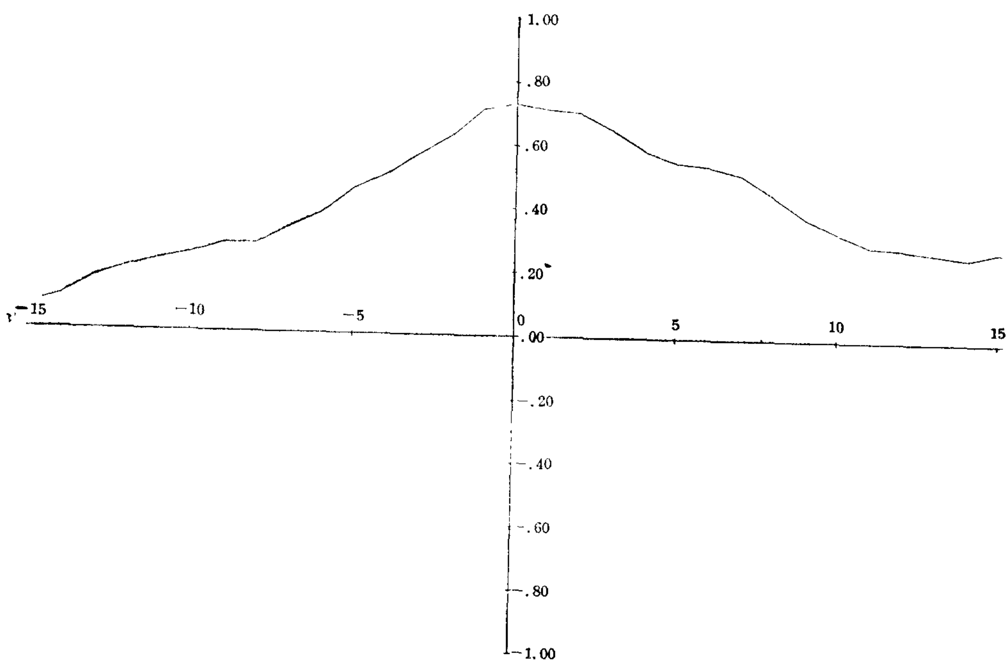


Fig.25. Pair E, ρ=0.99, N=50

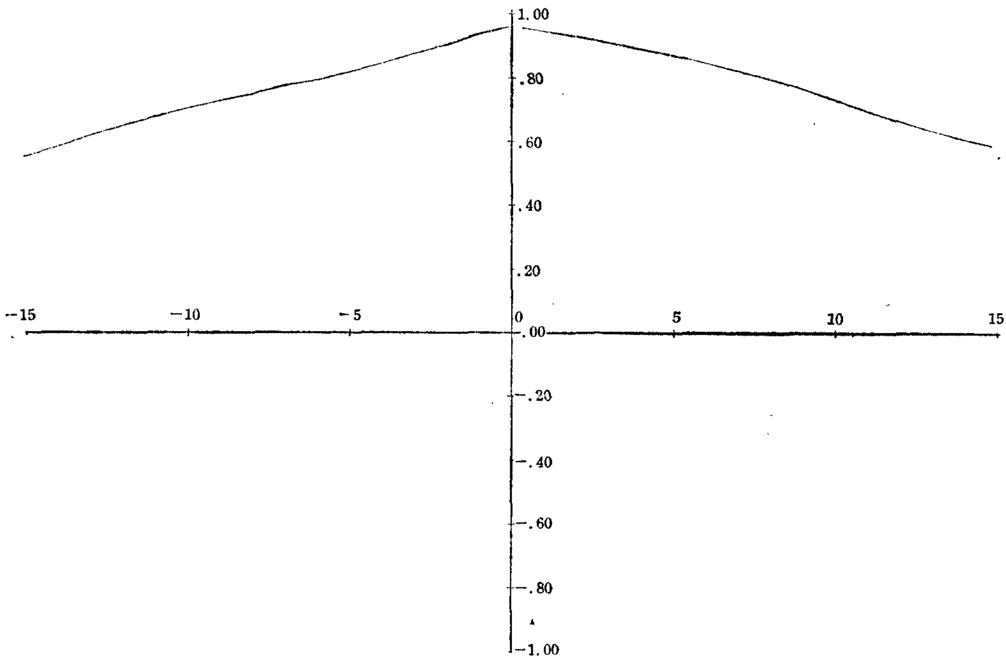


Fig.26. Pair E, $\rho=0.99$, $N=100$

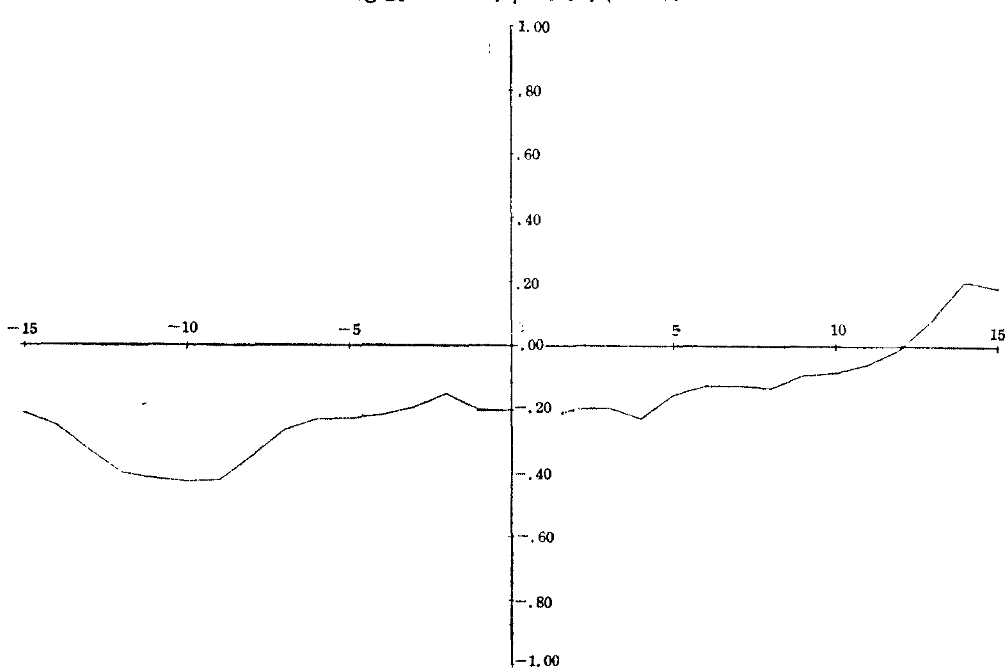


Fig.27. Pair F, $\rho=0.99$, $N=50$

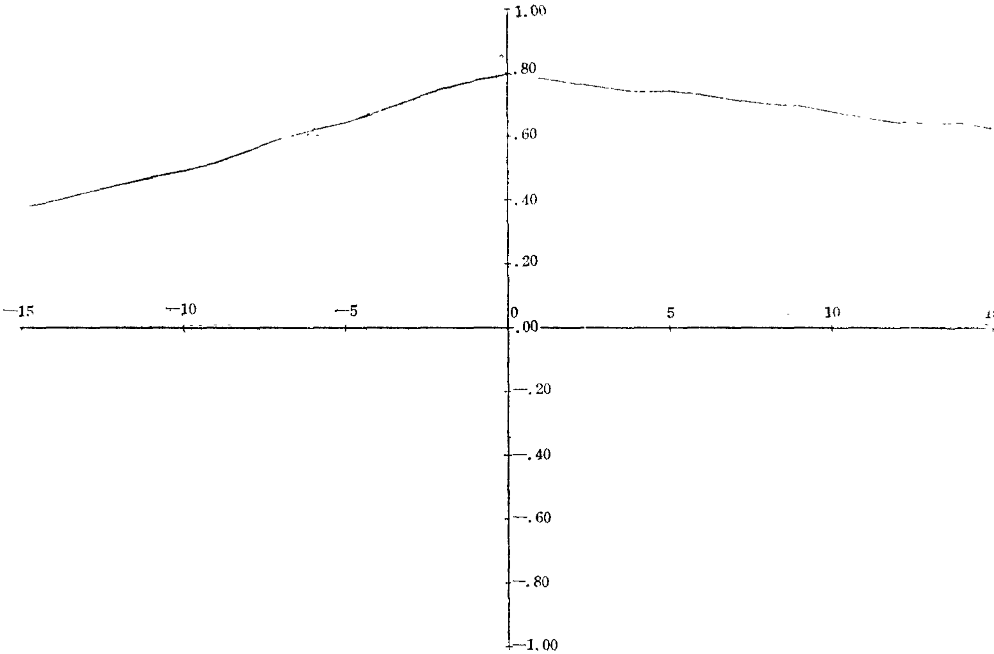


Fig.28. Pair F, $\rho=0.99$, $N=100$

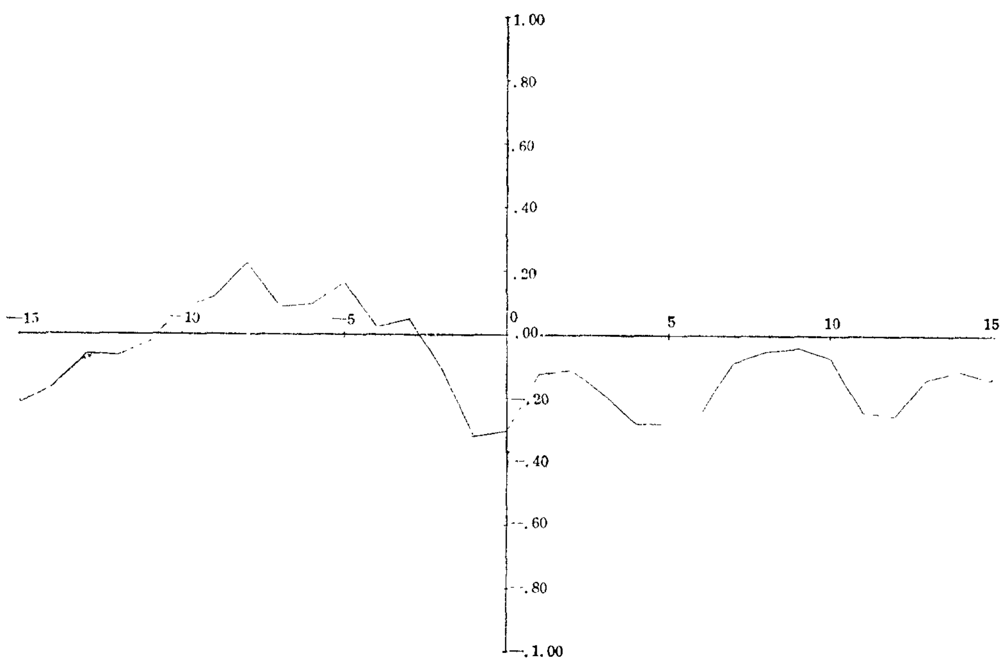


Fig.29. Pair A, $\rho=0.6$, $N=50$

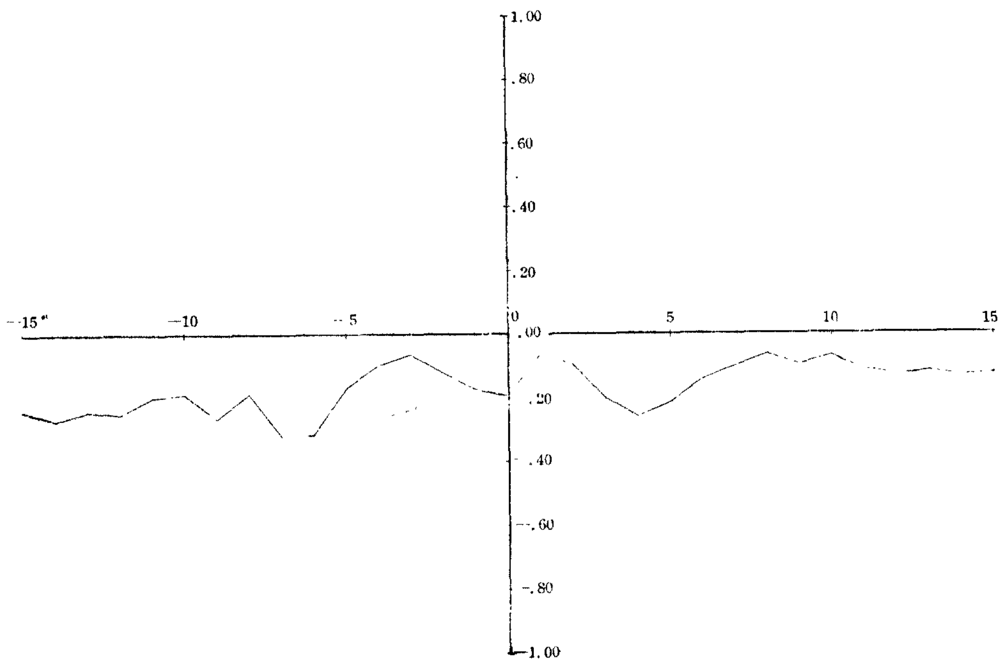


Fig.30. Pair A, $\rho=0.6$, $N=100$

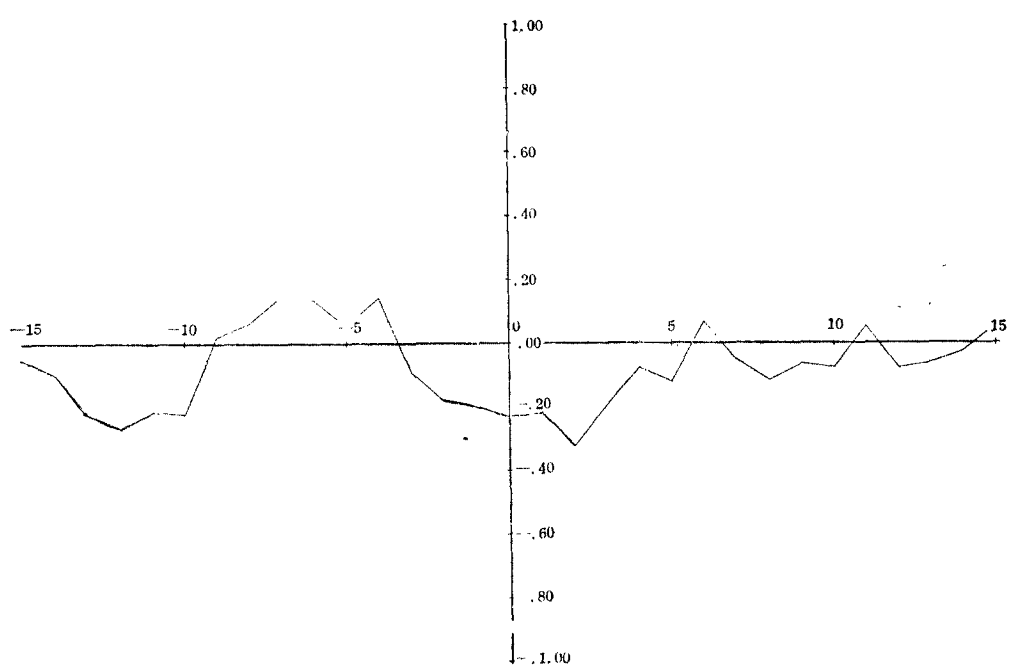


Fig.31. Pair B, $\rho=0.6$, $N=50$

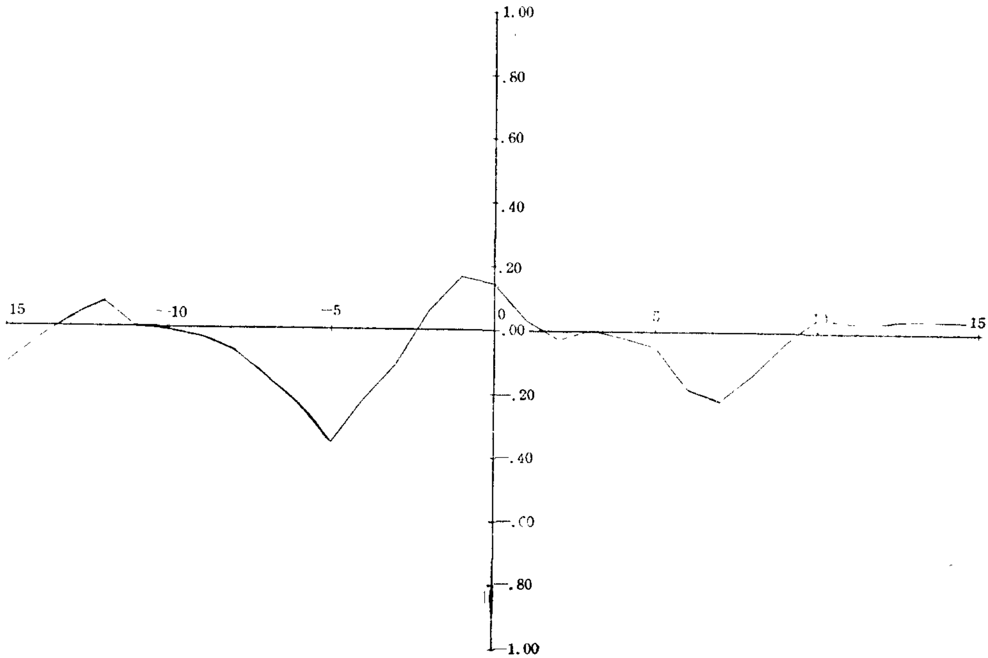


Fig.32. Pair B, $\rho=0.6$, $N=100$

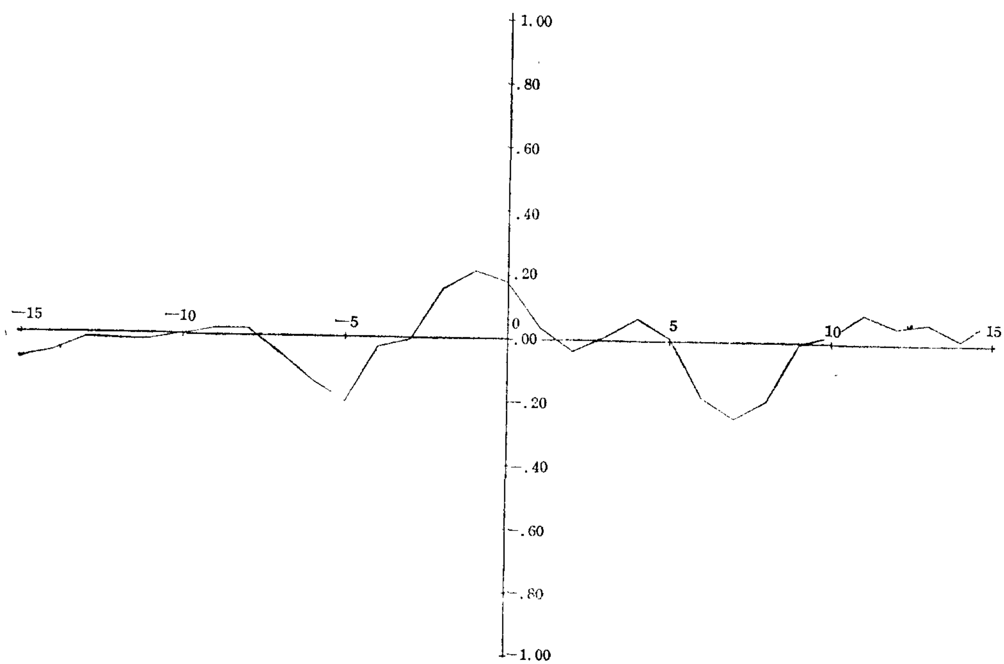


Fig.33. Pair C, $\rho=0.6$, $N=50$

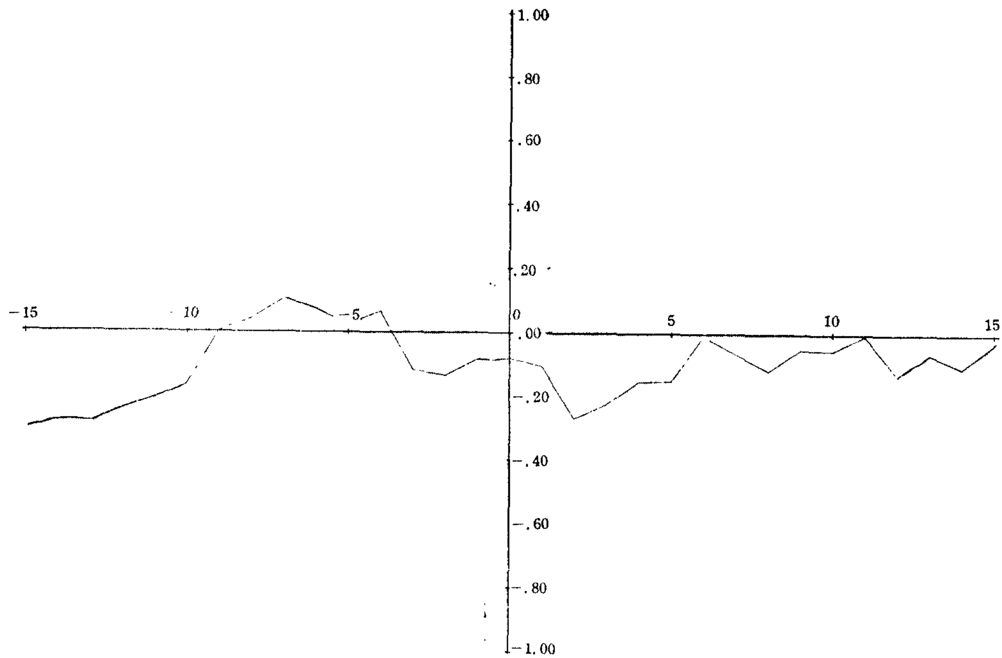


Fig.34. Pair C, $\rho=0.6$, $N=100$

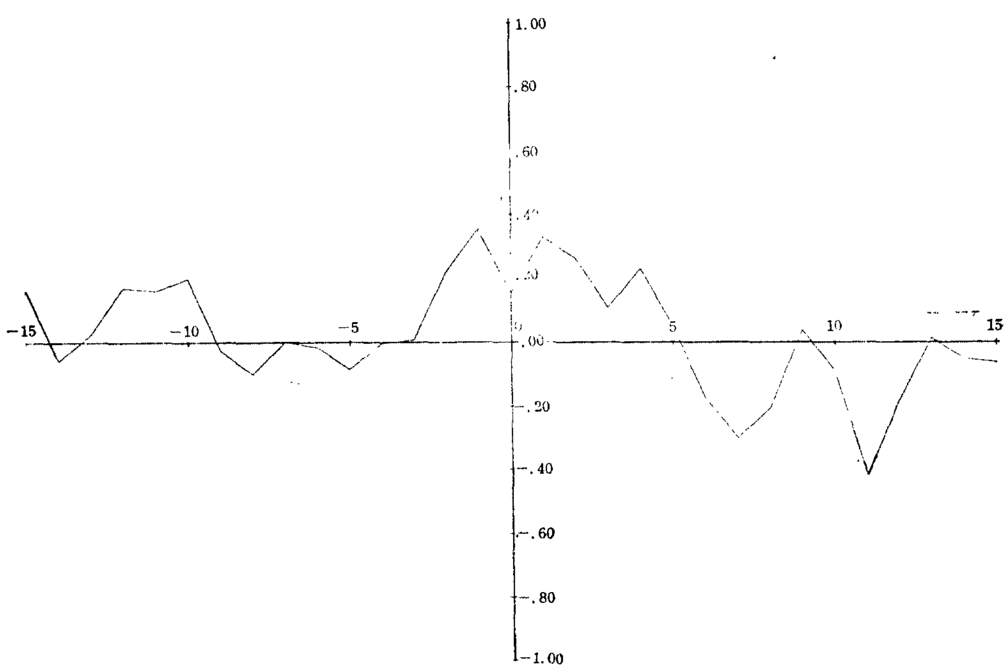


Fig.35. Pair D, $\rho=0.6$, $N=50$

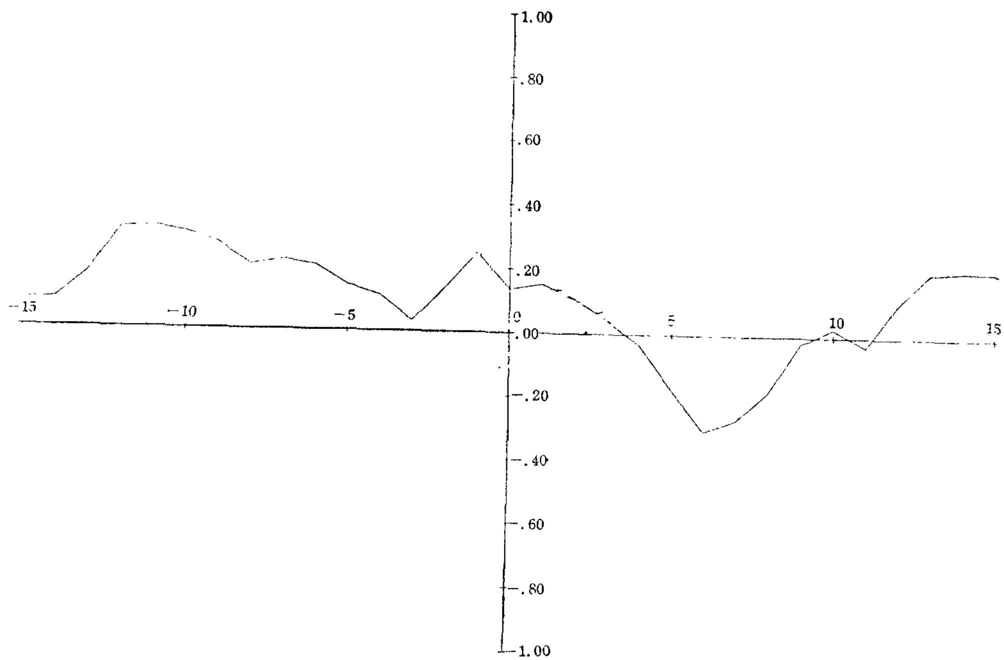


Fig.36. Pair D, $\rho=0.6$, $N=100$

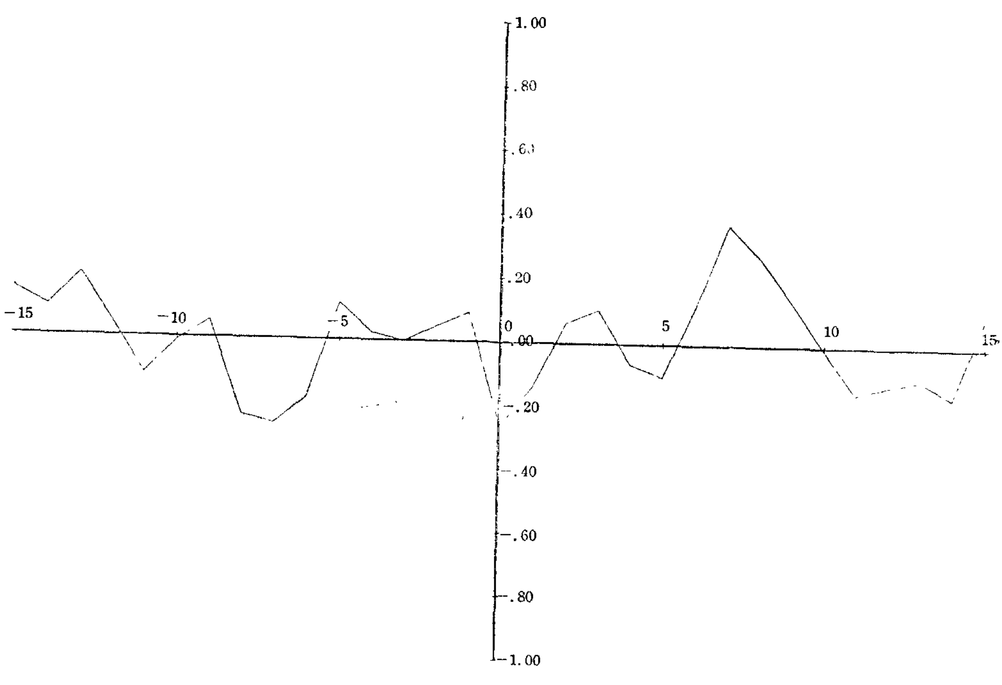


Fig.37. Pair E, $\rho=0.6$, $N=50$

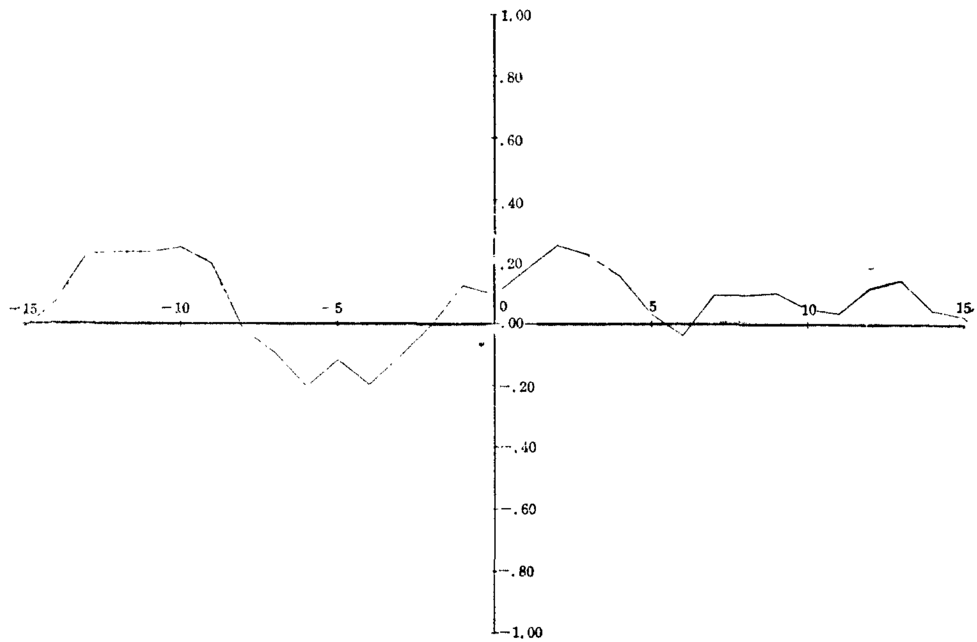


Fig.38. Pair E, ρ=0.6, N=100

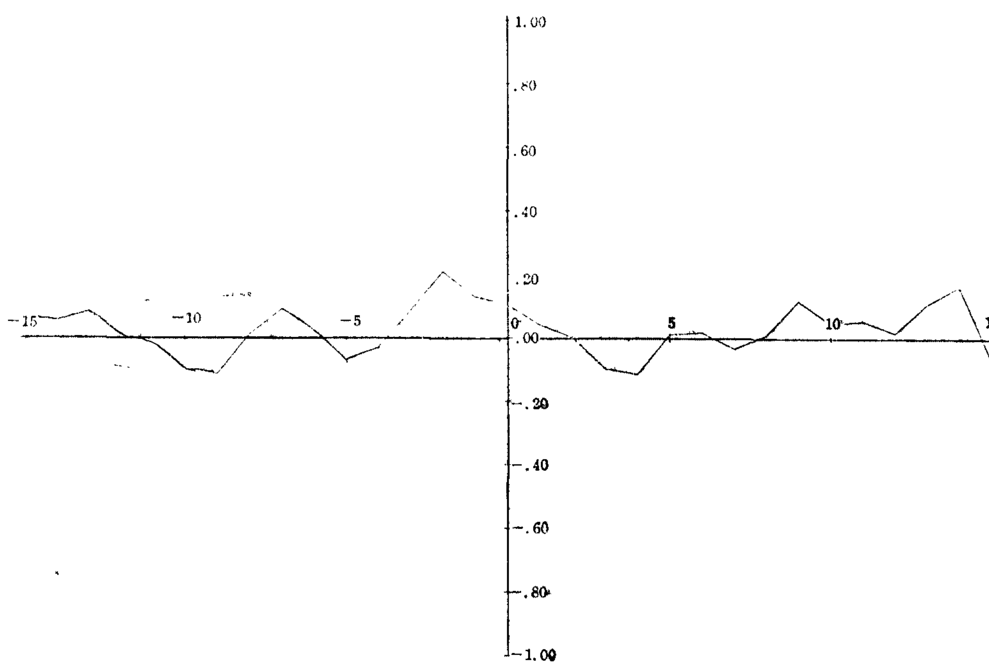


Fig.39. Pair F, ρ=0.6, N=50

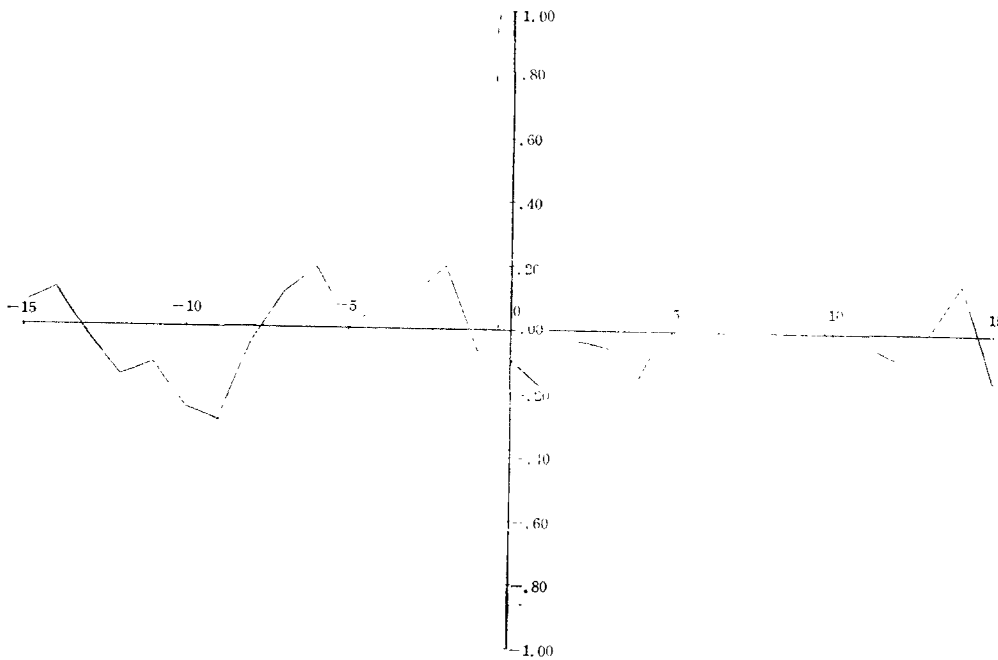


Fig.40. Pair F, $\rho=0.6$, $N=100$

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