A Note on the Complementarity and the Slutsky Equation

By Won-Key Kwon*

That all commodities demanded cannot be complementary when the choice among them corresponds to the optimization (maximization) of a quasi-concave utility function subject to a budget constraint is an important theorem in the theory of consumer demand. The theorem is usually proved using the Slutsky equation and the $(n+1)\times(n+1)$ matrix of differentials of the first-order conditions for utility maximization when there are n commodities.

In this note the theorem is proved in a more heuristic yet concise way exploiting the Slutsky equation and two properties of the demand function.

Property 1 (Single-valuedness and Continuity). The demand functions for each commodity derived from the first-order conditions of the utility maximization are single-valued in prices and income, continuous and twice differentiable:

$$X_i=D^i(p_1,p_2,\cdots,p_n,I)=D^i(p,I) \text{ for all } i=1,2,\cdots,n \tag{1}$$

where X_i is the i-th commodity; p_i is the price of the i-th commodity; p is the price vector ($p\geq 0$); and I is the money income.

Property 2 (Zero Homogeneity). The demand functions are homogeneous of degree zero in prices and income:

$$D^i(p,I)=D^i(tp,tI) \text{ for } t \text{ being constant and positive.} \tag{2}$$

Theorem. All commodities cannot be complements for one another.
Proof. From the Property 1

$$0=\sum_{j=1}^{n}\frac{\partial X_i}{\partial p_j}p_j+\frac{\partial X_i}{\partial I}I. \tag{3}$$

*The anthor is Senior Economist, Korea International Economic Institute.

From the Slutsky equation

$$\frac{\partial X_i}{\partial p_j}=\left(\frac{\partial X_i}{\partial p_j}\right)_{U=\text{constant}}-X_j\left(\frac{\partial X_i}{\partial I}\right)_{p=\text{constant.}} \tag{4}$$

Substituting equation (4) into equation (3)

$$0=\sum_{j=1}^{n}\left[\left(\frac{\partial X_i}{\partial p_j}\right)_{U=\text{constant}}-X_j\left(\frac{\partial X_i}{\partial I}\right)_{p=\text{constant}}\right]p_j+\frac{\partial X_i}{\partial I}I$$

$$=\sum_{j=1}^{n}\left[\left(\frac{\partial X_i}{\partial p_j}\right)_{U=\text{constant}}\right]p_j-\left[\left(\frac{\partial X_i}{\partial I}\right)_{p=\text{constant}}\right]\sum_{j=1}^{n}p_jX_j+\frac{\partial X_i}{\partial I}I$$

$$=\sum_{j=1}^{n}\left[\left(\frac{\partial X_i}{\partial p_j}\right)_{U=\text{constant}}\right]p_j-\left[\left(\frac{\partial X_i}{\partial I}\right)_{p=\text{constant}}\right]I+\frac{\partial X_i}{\partial I}I$$

$$=\sum_{j=1}^{n}p_j\left[\left(\frac{\partial X_i}{\partial p_j}\right)_{U=\text{constant}}\right] \tag{5}$$

or $\quad 0=\sum_{j=1}^{n}\eta_{ij}^{*}$ where $\eta_{ij}^{*}=\left[\left(\frac{\partial X_i}{\partial p_j}\right)_{U=\text{constant}}\right]\frac{p_j}{X_i}$

that is, the price elasticity corresponding to the pure substitution and $\underset{(i=j)}{\eta_{ij}^{*}}<0$. Therefore, all commodities can not be complementary. Q.E.D.

By using the similar method, we can also prove the result that all commodities cannot be inferior.

Proof. Taking the total differentiation of the budget equation

$$I=\sum_{i=1}^{n}p_iX_i$$

$$dI=\sum_{i=1}^{n}p_idX_i+\sum_{i=1}^{n}X_idp_i. \tag{6}$$

If we assume that prices are constant, i.e., $dp_i=0$ for all $i=1,2,\dots,n,$ in order to investigate the effect of change in money income on the consumption of each commodity, we get the following equation by dividing equation (6) by dI

$$1=\sum_{i=1}^{n}p_i\frac{dX_i}{dI}. \tag{7}$$

Therefore, $\frac{dX_i}{dI}$ cannot be negative for all i, that is, all goods are not inferior. Q.E.D.

References

[1] Henderson, J.M. and R.E. Quandt, *Microeconomic Theory*, 2nd ed., New York: McGraw-Hill, 1971, Chapter 2.

[2] Hicks, J.R., *Value and Capital*, 2nd ed., Oxford: Clarendon Press, 1946.

[3] Samuelson, P.A., *Foundations of Economic Analysis*, Cambridge, Mass.: Harvard, 1948.