On the Asymptotic Properties of the Inequality Constrained Generalized Least-Squares Estimation

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I. Introduction

There are growing demands to use prior and sample information for parameter estimation of a regression model. Several studies were done to meet such demands by Chipman-Rao [1], Theil [12, 13], Judge-Takayama [5], Liew [7, 8], Zellner [14], and Rothenberg [10].

The generalized least-squares estimation introduced by Zellner-Theil [15] and Jorgenson [4] reduces to indirect, two-stage and three-stage estimation depending on the identifiability condition and prior assumption on the covariance matrix of the residuals. This paper extends the generalized least-squares estimation so that it can cope with prior information and sample data, and it is called inequality constrained generalized least-squares (ICGLS) estimation. This paper also investigates the statistical properties of the ICGLS estimator and its covariance matrix in the case of a sufficiently large sample. Finally, it provides a numerical example of the ICGLS estimator.

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II. Model

Consider a complete system of p linear structural equations, all of which are identifiable, and suppose that the reduced form exists. Such a system of equations can be estimated by the generalized least-squares estimation model (Zellner-Theil [15], Jorgenson [4], and Madansky [9]).

$$\bar{X}'y=\bar{X}'Z\delta+\bar{X}'\varepsilon \qquad (1)$$

where y, δ, ε are vectors of jointly dependent variables, parameters, and residuals of the model respectively. X is a matrix of exogeneous variables of the model and $\bar{X}$ is $(I\times X)$ where I is an identity matrix and $\times$ is a Kronecker product.

The matrix Z is defined as below:

$$Z=\begin{bmatrix} Z_1 & & 0 \\ & \ddots & \\ 0 & & Z_p \end{bmatrix}$$

where $Z_i=[Y_i \vdots X_i]$ for $i=1,\cdots\cdots,p$.

The Y_i and X_i are matrices of explanatory endogeneous and exogeneous variables of ith structural equation.

We assume;

(i) X is a fixed matrix

(ii) X has a full rank

(iii) $E(\varepsilon)=o$

(iv) $V(\varepsilon)=\Omega\equiv(\Sigma\times I)$

where Σ is a symmetric positive definite matrix and o denotes a null vector.

With sample data, we wish to minimize the weighted sum of squares of the residuals in terms of d by restricting the conditions;

$Ad\geq c$ where d is an estimate of δ.(1)

The estimation problem can be formulated by the following primal-dual relations;

Primal

$$\underset{d}{\text{Min}}\ R=(1/2)(\bar{X}'y-\bar{X}'Zd)'\Omega^{-1}(\bar{X}'y-\bar{X}'Zd)$$

subject to

(1) Any mixed system can be converted to the inequality constraints; see Liew[7].

$$Ad \geq c$$

or

$$Ad - v = c \tag{2}$$

where v is a non-negative m-components surplus vector and d is otherwise unrestricted in sign.

Dual

$$\underset{\lambda}{\text{Max}}\ Q = c'\lambda + (1/2)(y'\bar{X}\Omega^{-1}\bar{X}'y - d'BZd)$$

subject to

$$A'\lambda + By = BZd \tag{3}$$

where

$$B \equiv Z'\bar{X}\Omega^{-1}\bar{X}' \tag{4}$$

and λ is an m-components dual vector and d is a solution to the primal problem.

The primal-dual relations reduce to the Dantzig-Cottle [2, 3] fundamental problem.

$$v = W\lambda + q \tag{5}$$

subject to

$$v'\lambda = 0,\ v \geq o \text{ and } \lambda \geq o \tag{6}$$

where

$$W = A(BZ)^{-1}A' \tag{7}$$

$$q = Ad^* - c \tag{8}$$

$$d^* = (BZ)^{-1}By \tag{9}$$

and d^* is the generalized least-squares estimator.(2)

Given q and W, the Dantzig-Cottle optimal solution becomes;

$$\begin{pmatrix} \bar{v} \\ \cdots \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} M_1 \\ \cdots \\ M_2 \end{pmatrix} \cdot q \tag{10}$$

where $\begin{pmatrix} \bar{v} \\ \cdots \\ \bar{\lambda} \end{pmatrix}$ is an m-components vector of the basic variables at optimal solution and

$$\begin{pmatrix} M_1 \\ \cdots \\ M_2 \end{pmatrix} = [I_1 \vdots -W_1]^{-1}$$

where $[I_1 \vdots -W_1]$ is $m \times m$ optimal basis.

By equations (3) and (9),

(2) See Zellner-Theil [15], Jorgenson [4] or Madansky [9].

$$d=d^*+(BZ)^{-1}(A_1^* \vdots A_2^*)' \begin{pmatrix} o \\ \cdots \\ \bar{\lambda} \end{pmatrix} \tag{11}$$

where $(A_1^*:A_2^*)'$ is a columnwise rearranged A matrix such that

$$A'\lambda=(A_1^*:A_2^*)' \begin{pmatrix} o \\ \cdots \\ \bar{\lambda} \end{pmatrix}.$$

d is an inequality constrained generalized least-squares (ICGLS) estimator vector.

By equations (8), (10), and (11),

$$d=(I+(BZ)^{-1} A_2^{*\prime}\cdot M_2\cdot A)\ d^*-(BZ)^{-1}A_2^{*\prime}\cdot M_2\cdot c \tag{12}$$

Since the optimal basis does not hold for all conceivable values of d^* except for the sufficiently large sample cases discussed in the next section, the covariance matrix of d is meaningful when the probability distribution of d^* is properly truncated. Such truncation is beyond the topic of this paper. Instead, the optimal basis obtained from a particular sample of d^* is imposed, and then an untruncated covariance matrix of d is derived.

$$V(d)\Big|_{\substack{\text{given}\\ \text{optimal}\\ \text{basis}}}=KV(d^*)K' \tag{13}$$

where

$$K=(I+(BZ)^{-1}A_2^{*\prime}M_2A).$$

III. Useful Lemmas

The ICGLS estimator (d) depends on the generalized least-squares estimator (d^*), covariance matrix of residuals (Σ) and the dual vector (λ) at the optimal basis; i. e.,

$$d=d(d^*,\ \Sigma,\lambda) \tag{14}$$

The following Lemmas are useful for deriving further results.

Lemma 1. If Σ is a diagonal matrix, and all elements of λ are equal to zero, then the ICGLS estimate vector d reduces to a two-stage least-squares estimator (d_2).

Proof. From equations (3), and if $\lambda=o$,

$$d=d^*=(Z'\bar{X}(\Sigma\times X'X)^{-1}\bar{X}'Z)^{-1}Z'\bar{X}(\Sigma\times X'X)^{-1}\bar{X}'y. \tag{15}$$

Since Σ is a diagonal matrix and $\bar{X}=(I\times X)$, d reduces to the following set of p equations

$$d_j=\sigma_{jj}(Z_j'X(X'X)^{-1}X'Z_j)^{-1}\left(\frac{1}{\sigma_{jj}}\right)(Z_j'X(X'X)^{-1}X'y_j)\quad(j=1,\cdots,p)$$

where σ_{jj}, Z_j and y_j are jth diagonal element of Σ, jth submatrix of Z and jth subvector of y respectively.

Since all σ_{jj}s are cancelled out, d_j can be stacked as below;

$$d=(Z'\bar{X}(I\times X'X)^{-1}\bar{X}'Z)^{-1}Z'\bar{X}(I\times X'X)^{-1}y=d_2. \tag{16}$$

We state the following Lemmas since analogous proofs were given elsewhere.(3)

Lemma 2. If Σ is a diagonal matrix and all elements of λ are strictly positive at the optimal solution, then the ICGLS estimate vector d reduces to an equality constrained two-stage least-squares estimate vector $\bar{d}_2$; i. e.,

$$d=d_2+(B^*Z)^{-1}A'(A(B^*Z)^{-1}A')^{-1}(c-Ad_2)=\bar{d}_2 \tag{17}$$

where

$$B^*\equiv Z'\bar{X}(I\times X'X)^{-1}\bar{X}'. \tag{18}$$

The covariance matrix of $\bar{d}_2$ becomes;

$$V(\bar{d}_2)=(B^*Z)^{-1}(I-A'(A(B^*Z)^{-1}A')^{-1}A(B^*Z)^{-1}). \tag{19}$$

Lemma 3. If Σ is a symmetric, positive definite matrix, and Σ is replaced by a two-stage estimate S of Σ, and if all elements of λ at the optimal basis are equal to zero, then the ICGLS estimate vector d reduces to a three-stage least-squares estimate vector d_3; i.e.,

$$d=(B^sZ)^{-1}B^sy=d_3 \tag{20}$$

where

$$B^s\equiv Z'\bar{X}(S\times X'X)^{-1}\bar{X}'. \tag{21}$$

Lemma 4. In Lemma 3, if all elements of λ at the optimal basis are strictly positive, the ICGLS estimate vector d reduces to an equality constrained three-stage least-squares estimate vector $\bar{d}_3$; i.e.,

$$d=d_3+(B^sZ)^{-1}A'(A(B^sZ)^{-1}A')^{-1}(c-Ad_3)=\bar{d}_3. \tag{22}$$

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(3) For proofs for Lemmas (2-5), see Jorgenson [4], Theil [13] and Liew [7, 8], and for proofs for Lemmas (6-8), see Jorgenson [4], Madansky [9], Rothenberg-Leenders [11], Zellner-Theil [15] and Theil [13].

The covariance matrix of $\bar{d}_3$ becomes;

$$V(\bar{d}_3)=(B^sZ)^{-1}(I-A'(A(B^s Z)^{-1} A')^{-1} A(B^sZ)^{-1}). \tag{23}$$

Lemma 5. If the model (1) is exactly identifiable, and Σ is a diagonal matrix, the ICGLS estimate vector d reduces to indirect least-squares estimate vector (d_1) when the optimal $\lambda=o$, and the d reduces to an equality constrained indirect least-squares estimate vector ($\bar{d}_1$) when the optimal $\lambda\rangle\rangle o$.

Lemma 6. Under usual assumptions, two-stage least-squares estimates d_2 and three-stage least-squares estimates d_3 are asymptotically unbiased and consistent estimates of δ.

Lemma 7. Under certain assumptions, equality constrained weighted least-squares estimates are best linear unbiased regression estimates under the linear prior restriction.

Lemma 8. The two-stage and three-stage least-squares estimates (d_2 and d_3) are the weighted least-squares estimates.

Lemma 9. The equality constrained two- and three-stage least-squares estimates ($\bar{d}_2$ and $\bar{d}_3$) are the equality constrained weighted least-squares estimates.

IV. Asymptotic Properties of the ICGLS Estimator

To show the asymptotic properties of the ICGLS estimates, we consider two cases; (1) all true parameters are unbounded (i. e., $A\delta\rangle\rangle c$) and (2) some parameters are unbounded while the others are bounded (i. e., $A_1\delta\rangle\rangle c_1$ and $A_2\delta=c_2$).

Theorem 1. If the prior belief ($A\delta\rangle\rangle c$) is correct, then there exists a sufficiently large sample $n\geq n_0$ which makes all elements of dual vector λ zero at the optimal solution.

Proof. The solutions to the Dantzig-Cottle system ($v=W\lambda+q$, $v'\lambda=0$, $\lambda\geq o$ and $v\geq o$) imply that when $q\rangle\rangle 0$, all elements of λ become zero. To complete the proof, we need to show that there exists sufficiently large sample $n\geq n_0$ which makes all elements of q vector positive.

By relations (8), (9), (4) and (1)

$$q_n = A\delta - c + A\cdot\phi_n \tag{24}$$

where

$$\phi_n = (n^{-1}(Z'\bar{X})(\Sigma\times n^{-1}(X'X))^{-1}n^{-1}(\bar{X}'Z))^{-1} \cdot n^{-1}(Z'\bar{X})(\Sigma\times n^{-1}(X'X))^{-1}\, n^{-1}\bar{X}'\varepsilon. \tag{25}$$

Subscript n denotes the sample size. With the usual assumptions of the asymptotic sampling theory, we can show that ϕ_n vanishes as the sample size increases sufficiently large.[4] If the prior belief is correct, $A\delta - c \gg o$ and it remains constant whereas $A\phi_n$ is getting smaller as the sample size n increases. Therefore, there exists a sufficiently large sample $n \geq n_0$ which makes $q_n \gg o$.

Corollary 1. If the prior belief ($A\delta - c \gg o$) is correct and Σ is a diagonal matrix, then the ICGLS estimate vector d becomes an asymptotically unbiased and consistent estimator of δ. In this case, the untruncated covariance matrix of d shares the same asymptotic properties of the covariance matrix of two-stage least-squares estimates.

Proof. Theorem 1 states that there is a sufficiently large sample $n \geq n_0$ which makes $q_n \gg o$ and $\lambda = o$. Lemma 1 states that if Σ is a diagonal matrix and $\lambda = o$, the ICGLS estimate vector d reduces to the two-stage least-squares estimate vector d_2 which is asymptotically unbiased and consistent estimator of δ by Lemma 6.

Corollary 2. If the prior belief ($A\delta - c \gg o$) is correct and Σ is replaced by a consistent estimate S obtained from the two-stage estimates, then ICGLS estimate vector d shares the same asymptotic properties of three-stage least-squares estimate vector d_3. In this case, the untruncated covariance matrix of d shares the same asymptotic properties of the covariance matrix of d_3.

Proof. By Theorem 1 and Lemma 3, we can show that there exists a sufficiently large sample $n \geq n_0$ which reduces d to d_3.

Next we consider the case where some parameters are bounded and some are unbounded (i.e., $A_1\delta \gg c_1$ and $A_2\delta = c_2$).

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(4) See Jorgenson [4].

Theorem 2. If the prior belief ($A_1\delta \rangle\rangle c_1$ and $A_2\delta = c_2$) is correct, then there exists a sufficiently large sample $n \geq n_0$ such that the ICGLS estimate vector d reduces to an equality constrained generalized least-squares (ECGLS) estimators ($\bar{d}$).

Proof. In this case, the sample estimates (d) have two constraints; (1) $A_1 d \geq c_1$ and (2) $A_2 d = c_2$. The equations (2) and (3) are partitioned as below:

$$A_1 d - v_1 = c_1 \tag{26}$$

$$A_2 d = c_2 \tag{27}$$

$$(A_1' : A_2') \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + By = BZd \tag{28}$$

where

$$B \equiv Z'\bar{X}(\Sigma \times X'X)^{-1}\bar{X}'.$$

By equations (27) and (28),

$$\lambda_2 = -W_{22}^{-1}\ W_{21} \cdot \lambda_1 - W_{22}^{-1}(A_2(BZ)^{-1}By - c_2) \tag{29}$$

where

$$W_{ij} \equiv A_i(BZ)^{-1}\ A_j' \quad i, j = 1, 2.$$

By equations (26) and (28),

$$v_1 = W_{11}\lambda_1 + W_{12}\lambda_2 + A_1(BZ)^{-1}\ By - c_1. \tag{30}$$

By equations (29) and (30),

$$v_1 = M^*\lambda_1 + q^* \tag{31}$$

where

$$M^* = W_{11} - W_{12} \cdot W_{22}^{-1} \cdot W_{21} \tag{32}$$

$$q^* = -W_{12} \cdot W_{22}^{-1} \cdot (A_2(BZ)^{-1}\ By - c_2) + (A_1(BZ)^{-1}\ By - c_1). \tag{33}$$

Since $y = Z\delta + \varepsilon$,

$$(BZ)^{-1}By = \delta + (BZ)^{-1}B\varepsilon. \tag{34}$$

Let q_n^* be the q^* on sample size n. By equations (33) and (34), the following relation is evident.

$$b_n^* = (A_1\delta - c_1) - W_{12}W_{22}^{-1}(A_2\delta - c_2) + (A_1 - W_{12}W_{22}^{-1}A_2)(B_nZ_n)^{-1}B_n\varepsilon \tag{35}$$

where

$$(B_nZ_n)^{-1}B_n\varepsilon \equiv (n^{-1}Z'\bar{X}(\Sigma\times n^{-1}X'X)^{-1}\ n^{-1}\bar{X}'Z)^{-1} \cdot n^{-1}Z'\bar{X}(\Sigma\times n^{-1}X'X)^{-1}\ n^{-1}\bar{X}'\varepsilon. \qquad (36)$$

The second term of equation (35) vanishes because of the prior information (i. e., $A_2\delta - c_2 = o$) and the third term is getting smaller under usual assumptions of the asymptotic sampling theory as sample size is getting larger.[5] The first term is positive and remains constant as sample size increases. Therefore, there exists a sufficiently large sample $n \geq n_0$ which makes $q_n^* \gg o$. The subscript n hereafter is deleted for notational simplification.

When $q^* \gg o$, the Dantzig-Cottle solutions are;

$$v_1 = q^* \gg o \text{ and } \lambda_1 = o. \qquad (37)$$

By equations (28), (29) and (37),

$$d = d^* + (BZ)^{-1}\ A'_2(A_2(BZ)^{-1}A'_2)^{-1}\ (c_2 - A_2d^*) = \bar{d} \qquad (38)$$

where

$$d^* = (BZ)^{-1}By.$$

Corollary 3. If the prior belief $(A_1\delta \gg c_1$ and $A_2\delta = c_2)$ is correct and Σ is a diagonal matrix, then there exists a sufficiently large sample $n \geq n_0$ such that the ICGLS estimate vector d reduces to an equality constrained two-stage least-squares estimate vector $\bar{d}_2$. In this case, the ICGLS estimate d is the best linear unbiased regression under the linear prior restriction and the untruncated covariance of d shares same statistical properties of the covariance matrix of $(\bar{d}_2)$.

Proof. It follows immediately from the proofs of Theorem 2, Lemmas 1, 7, 8 and 9.

Corollary 4. In the Corollary 3, if Σ is replaced by a consistent estimate S from two-stage least-squares estimates, then there exists a sufficiently large sample $n \geq n_0$ such that the ICGLS estimate vector d reduces to an equality constrained three-stage least-squares estimate vector $\bar{d}_3$. Therefore, the ICGLS estimate vector d and its covariance matrix $(V(d))$ shares the

(5) See Jorgenson [4].

same asymptotic properties of equality constrained three-stage least-squares estimate vector $\bar{d}_3$ and their covariance matrix $(V(\bar{d}_3))$.

Proof. It follows immediately from the proofs of Theorem 2, Lemmas 3, 7 and 8.

V. A Numerical Example

Suppose we have the following new demand and supply model for electricity in the United States.

Demand Equation

$$\log q_i = \log\alpha_0 + \alpha_1 \log p_i + \alpha_2 \log \pi_i + \alpha_3 \log y_i + \varepsilon_{1i}$$

Supply Equation

$$\log p_i = \log\beta_0 + \beta_1 \log q_i + \beta_2 \log z_i + \varepsilon_{2i}$$

where

q_i is the new demand for electricity (KWH) for ith state in 1970.

p_i is the price of electricity ($) per KWH for ith state in 1970.

π_i is the price of natural gas per Therm for ith state in 1970.

y_i is the expenditure on new demands for electricity and natural gas (i.e., $y_i = p_i q_i + \pi_i G_i$) where g_i is the new demand for natural gas for ith state in 1970.

z_i is a proxy to the capacity for the new supply of electricity. Installed electricity generating capacity (million KWH) was used in the computation as a proxy to the capacity. ε_{1i} and ε_{2i} are residuals.

To maintain structural consistency, the following restrictions were made:

Demand Equation

(i) $\alpha_1 \leq 0$ (negative demand price elasticity)

(ii) $\alpha_2 \geq 0$ (positive cross-elasticity)

(iii) $\alpha_3 \geq 0$ (positive income elasticity)

(iv) $\alpha_1 + \alpha_2 + \alpha_3 = 0$ (the homogeneity condition)

Supply Equation

(i) $\beta_1 \geq 0$ (positive supply price elasticity)

(ii) $\beta_2 \leq 0$ (negative capacity elasticity with respect to price)

Utilizing 51 observations, we calculate parameters of the electricity model. We obtain the following empirical results.(6)

Demand Equation

Parameters	Two-Stage Estimate		ICGLS Estimate*	
	Estimate	Std. Error	Estimate	Std. Error
log α_0	−0.9865	0.5046	−0.2910	0.5097
α_1	−1.3652	0.1366	−1.1812	0.1269
α_2	0.2219	0.0499	0.2325	0.0532
α_3	0.9417	0.0159	0.9487	0.0163

Supply Equation

Parameters	ICGLS and Two-Stage Estimate**	
	Estimate	Std. Error
log β_0	−6.3088	1.1045
β_1	0.3482	0.1504
β_2	−0.3818	0.1442

The Estimated Variance-Covariance Matrix of the ICGLS Estimates

$$\begin{pmatrix} .25983 & .06279 & .00989 & .00071 \\ & .01612 & .00130 & .00041 \\ & & .00283 & .00024 \\ & & & .00027 \end{pmatrix}$$

* ICGLS estimate when Σ is a diagonal matrix.

** None of the constrained is bounded, the ICGLS estimate is same as the two-stage estimate.

Sources of Data: The data were collected from the following sources: (a) U.S. Department of Commerce, *Statistical Abstract of the U. S.,* Washington, D. C.: U.S. Government Printing Office, 1971, 1972 and 1974. (b) U. S. Federal Power Commission, *Statistics of Privately-Owned Electric Utilities in the United States,* Washington, D. C.: U.S. Government Printing Office, 1969 and 1970. (c) U. S. Federal Power Commission, *Statistics of Publicly-Owned Electric Utilities in the United States,* Washington, D. C.: U. S. Government Printing Office, 1969 and 1970.

The new demand for electricity (q_i) and for natural gas (g_i) was estimated as below:

$$q_i(1970) = Q_i(1970) - 0.9\ Q_i(1969)$$
$$g_i(1970) = G_i(1970) - 0.9\ G_i(1969)$$

where Q_i (1970) is per capita electricity consumption in ith state during 1970 and G_i (1970) is per capita natural gas consumption in ith state during 1970.

References

[1] Chipman, J. S. and M. M. Rao, "The Treatment of Linear Restrictions in Regres-

(6) The author constructed ICGLS program when Σ is a diagonal matrix by utilizing the IBM scientific subroutines and from the Lemke program obtained from Operation Research Center, the University of California, Berkeley.

sion Analysis," *Econometrica*, Vol. 32 No. 1-2 (January-April 1964), pp. 198-209.

〔2〕 Cottle, R. W. and G. B. Dantzig, "Complementary Pivot Theory of Mathematical Programming," *Lectures in Applied Mathematics*, Vol. 11, Stanford University.

〔3〕 Dantzig, G. B. and R. W. Cottle, "Positive (semi-)Definite Matrices and Mathematical Programming," *Nonlinear Programming*, J. Abadie, ed., North-Holland, Amsterdam, 1967, pp. 55-73.

〔4〕 Jorgenson, D. W., *Lecture Notes: Economics* 241, Part IV: *The Multivariate Structural Model*, Berkeley, 1962.

〔5〕 Judge, G. G. and T. Takayama, "Inequality Restrictions in Regression Analysis," *Journal of American Statistical Association*, Vol. 61 (1966), pp. 166-181.

〔6〕 Lemke, C. E., "A Method of Solution for Quadratic Programs," *Management*, August 1962, pp. 442-53.

〔7〕 Liew, C. K., "Inequality Constrained Least-Squares Estimation," *The Journal of American Statistical Association*, 72 (September 1976).

〔8〕 Liew, C. K., "A Two-Stage Least-Squares Estimation with Inequality Restrictions on Parameters," *The Review of Economics and Statistics*, LVII (May 1976).

〔9〕 Madansky, A., "On the Efficiency of Three-Stage Least-Squares Estimation," *Econometrica*, Vol. 32 No. 1-2 (January-April 1964).

〔10〕 Rothenberg, T. J., *Efficient Estimation with A Priori Information*, Cowles Foundation Monograph, No. 23 (1973), pp. 132-153.

〔11〕 Rothenberg, T. J. and C. T. Leenders, "Efficient Estimation of Simultaneous Equation Systems," *Econometrica*, Vol. 32 No. 1-2 (January-April 1964).

〔12〕 Theil, H., "On the Use of Incomplete Prior Information in Regression Analysis," *Journal of the American Statistical Association*, Vol. 58 (June 1963), pp. 401-14.

〔13〕 Theil, H., *Economic Forecasts and Policy*, North Holland, 1970, pp. 536-538.

〔14〕 Zellner, A., *Introduction to Bayesian Inference in Econometrics*, New York: Wiley, 1971.

〔15〕 Zellner, A. and H. Theil, "Three-Stage Least-Squares: Simultaneous Estimation of Simultaneous Equations," *Econometrica*, Vol. 30 (1962), pp. 54-78.