Dynamic Adjustment of Expectational Equilibria

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I. Introduction

A rational expectations equilibrium is an economic equilibrium where agents' models or theories of what the equilibrium will be are fulfilled by the actual allocation generated when agents use these models in their optimizing behaviour. The main purpose of this paper is to investigate a learning process through which a sequence of expectational equilibria converges to a rational expectations equilibrium, and to support the notion of a self-fulfilling equilibrium when the economy is subject to uncertainty. Previous study on this area undertaken by Kihlstrom and Mirman [13] is from a Bayesian point of view. They show that when the price process is a stationary stochastic process, the outside Bayesian observer can form the same price expectations as insiders who know the structural parameter of the economy. Since the relevant state of nature and the price are independent of the actions of the outside observers, their learning does not influence the distribution of the price process. Jordan [12] takes a different approach. He hypothesizes that there are intermediate stages at which each agent learns about the functional relation between the sequence of equilibrium prices and the corresponding states, and that no intermediate temporary equilibria is required to be consummate. In this

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paper, we assume that all economic agents participate in the market even if they do not have a perfect model of how the market clearing price is related to the particular event obtained in each period.

We define an expectational equilibrium as an equilibrium that results when the agents have an imperfect model of price formation process and when their model is not controverted by the observation of an equilibrium. If every agent improves his model over time, the price process will not, in general, be a stationary process. Although the capacity to interpret a price signal differs among agents, we show that the price system can still convey all the information available in the economy. Until the learning process is finished, informed agents can collect a rent to their information endowment. This confirms the idea that there is an investment aspect in the acquisition of costly information. We show that in the limit of dynamic adjustment process, a rational expectations equilibrium persist over time and the price system becomes a source of externalities (Green [7]). In section II, the economy and the model are described to present an existence theorem. In section III, we introduce an imperfect model of agents and show how the learning process can proceed. In the final section the uniqueness of an expectational equilibrium as well as the convergence of our learning process is proved.

II. A Model of Rational Expectations Equilibrium

1. The Economy and the Model

We consider a perfectly competitive, pure exchange economy in a stochastic environment. The economy consists of a finite set $\{1, 2, \ldots, T\}$ of agents. Each agent is uncertain about the state of nature until after the exchange operation is completed. We assume that the economy lasts for two periods. In the first period, agents exchange their endowments but do not consume. In the second period, no exchange takes place and each agent consumes what he obtained in the first period. Every agent is assumed to have a private information source about his uncertain environment. We call it a signal and we assume that every agent's signal is correlated with the payoff relevant state of the environment. Formally, let Ω be the set of all possible states. Associated with the set Ω is a specified sigma-field of events, or measurable subsets, denoted by S. We assume that every agent has a common prior probability measure P on the measurable space (Ω, S). The signal of agent i, $\tilde{y}_i$, is a random variable defined on the probability space (Ω, S, P). Let $\boldsymbol{B}(\tilde{y}_i)$ be a sigma-field of events generated by $\tilde{y}_i$. Then we have $\boldsymbol{B}(\tilde{y}_i) \subset S$. Equivalently, an event $B \varepsilon S$ is a member of $\boldsymbol{B}(\tilde{y}_i)$ if and only if agent i knows whether or not the true state that actually obtains belongs to B upon receiving his signal $\tilde{y}_i$. The characteristics of a representative agent i are described as a triple $(\gtrsim_i(\omega),\ \bar{x}_i,\ \tilde{y}_i)$, where $\gtrsim_i(\omega)$ denotes his statedependent preference ordering, an ordering that can be represented by a real valued function $u_i(\cdot;\ \omega)$ for each state ω; $\bar{x}_i$ is his endowment, which we assume is constant over all states; $\tilde{y}_i$ is his signal. The consumption possibility set X_i of every agent is assumed to be $R_+{}^N$, the non-negative orthant of the N-dimensional Euclidean space, where N denotes the number of commodities that are traded. Any particular element of $R_+{}^N$ is denoted by x_i, which is the consumption of a representative agent i. The following assumption is maintained throughout the paper.

Assumption 2. 1 : Assume $u_i : R_+{}^N \times \Omega \rightarrow R$ is continuous in $x_i \ \varepsilon\ R_+{}^N$ for each ω and S-measurable for each x_i. We also assume that $u_i(x_i;\ \omega)$ is strictly concave, increasing in x_i and bounded. Further, $u_i(x_i;\ \omega)$ is differentiable in x_i almost surely (a.s.).

Let $\tilde{y}_i : (\Omega, S, P) \rightarrow R^M$ for each i and let Y_i denote the range of $\tilde{y}_i$ in R^M, the M-dimensional Euclidean space. Define the space Y of a joint signal, $(\tilde{y}_i)_{i=1}{}^T$, by the cross product of all the Y_i 's, i.e., $Y = Y_1 \times \ldots \times Y_T$. The realization of each signal $\tilde{y}_i$ is denoted by y_i. Define $\tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_T)$ and $\tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_T)$. We notice that each signal $\tilde{y}_i$ induces a probability measure ν on Y by the operation $P \circ \tilde{y}^{-1}$. In other words, for any Borel subset A of Y, we have $\nu(A) = P(\tilde{y}^{-1}(A))$.

Each agent is maximizing his expected utility, conditional on his information, including information that is transmitted by the market price, subject to his budget constraint. Formally, agent i chooses his consumption vector $\tilde{x}_i(y)$ in the event $B(y) = \{\omega \varepsilon \Omega \mid \tilde{y}(\omega) = y\}$ by

$$
\begin{aligned}
&\text{maximizing} \quad && E\{u_i(\tilde{x}_i(y);\ \omega) \mid \tilde{y}_i(\omega) = y_i,\ \ \tilde{p}(y) = p\} \\
&\text{subject to} \quad && \tilde{p}(y)\tilde{x}_i(y) \leq \tilde{p}(y)\bar{x}_i.
\end{aligned}
$$

The price $\tilde{p}(\cdot)$ is a vector-valued random variable defined on the space of joint signals Y,

$$\tilde{p}(\cdot):(Y,\boldsymbol{y})\rightarrow(S_N,\ \sigma(S_N)),$$

where S_N is the unit price simplex in R_+^N; $\boldsymbol{y}$ and $\sigma(S_N)$ are the Borel-fields defined on Y and S_N respectively. The consumption vector $\tilde{x}_i(\cdot)$ is also a vector-valued random variable and $\tilde{x}_i(\cdot):(Y,\boldsymbol{y})\rightarrow R_+^N$. These notations embody the idea that the market clearing price as well as the consumption choice does not depend on the information that is not available in the economy. Further, strict concavity of $u_i(x_i;\ \omega)$ in x_i excludes the possibility that the consumption vector chosen in an equilibrium can provide additional information that is not contained in the price (Kreps [14]). Let $\boldsymbol{F}_i=\boldsymbol{B}(\tilde{y}_i)\wedge\boldsymbol{B}(\tilde{p})$, the smallest σ-field of events that contains $\boldsymbol{B}(\tilde{y}_i)$ and $\boldsymbol{B}(\tilde{p})$, where $\boldsymbol{B}(\tilde{p})$ is the σ-field of events induced by $\tilde{p}$ on Ω. Then

$$\begin{array}{ll}\text{maximizing} & E\{u_i(\tilde{x}_i(y);\omega)\,|\,\tilde{y}_i(\omega)=y_i,\ \tilde{p}(y)=p\}\\ \text{subject to} & \tilde{p}(y)\tilde{x}_i(y)\leq\tilde{p}(y)\bar{x}_i\end{array}$$

is equivalent to

$$\begin{array}{ll}\text{maximizing} & E\{u_i(x_i;\omega)\,|\,\boldsymbol{F}_i\}(\omega)\\ \text{subject to} & \tilde{p}(y)x_i(y)\leq\tilde{p}(y)\bar{x}_i\end{array}$$

for any state $\omega\varepsilon B(y)$. We always take regular versions of conditional expectations so that the consumption vector of each agent as well as the resulting market clearing price is well defined up to P-equivalence or except on a P-null set (Kreps [14]).

The demand function of agent i is denoted as $x_i(\boldsymbol{F}_i,p)$, given the price $\tilde{p}(\cdot)$. An equilibrium price $\tilde{p}(\cdot)$ is such that

$$\sum_{i=1}^{T}x_i(\boldsymbol{F}_i,p)\leq\sum_{i=1}^{T}\bar{x}_i \text{ for all } y\varepsilon Y$$

except on a ν-null set, where $\tilde{p}(y)=p$.

We call $(x_i(\boldsymbol{F}_i,p),\tilde{p})_{i=1}{}^T$ a *rational expectations equilibrium* for this economy. This notion of equilibrium comes from the concept of stationary equilibrium. Every agent eventually learns the true joint distribution of $(\tilde{y}_i,\tilde{p})$ if this economy is repeated over time. Through a sequence of temporary equilibria, agents might have a chance to know what the function $\tilde{p}(\cdot)$ looks like. This is a highly idealized version of the notion that price conveys information and that no learning must be possible at an equilibrium.

A rational expectations equilibrium price $\tilde{p}$ is called *fully revealing* if $\boldsymbol{B}(\tilde{p})=\boldsymbol{B}(\tilde{y})$, i.e., if price conveys all the information held by agents. The corresponding equilibrium is called a full-information equilibrium. A rational expectations equilibrium price need not necessarily be fully revealing in our sense. For example, an equilibrium price may just provide a sufficient statistic for a joint signal in determining the optimal demand of each agent (Grossman [10]). When the cardinality of a signal space is infinite, the concept of a negligible set in the space of economies will be sensitive to the choice of topology as has been illustrated by Green [8], and Radner and Jordan [18], where the *openness* of non-existence of a rational expectations equilibrium depends on the choice of topology in the space of economies. In the following, we provide a set of sufficient conditions for the existence of a fully revealing equilibrium.

2. Existence Theorem

When price conveys full information, i.e., when $\boldsymbol{B}(\tilde{p})=\boldsymbol{B}(\tilde{y})$, let us denote $\boldsymbol{F}_i=\boldsymbol{B}(\tilde{y}_i)\wedge\boldsymbol{B}(\tilde{p})$ by $\boldsymbol{F}$ for all i. The following proposition is useful to prove that an equilibrium for each information-event in $\boldsymbol{F}$ is virtually equivalent to the Arrow-Debreu equilibrium.

Proposition 2.1 : Let $\boldsymbol{F}$ be any sub-sigma-field of events such that $\boldsymbol{F}\subset S$. Then $E\{u_i(x_i;\omega)|\boldsymbol{F}\}$ is continuous in $x_i \in R^N$ for almost all $\omega \in \Omega$ and $\boldsymbol{F}$-measurable for each x_i. Also, $E\{u_i(x_i;\omega)|\boldsymbol{F}\}$ is strictly concave, increasing in x_i and bounded from above. It is differentiable almost surely.

Proof: Using the Lebesque Dominated Convergence Theorem on conditional expectation (Chung [3]), we can easily prove continuity and differentiability of $E\{u_i(x_i;\omega)|\boldsymbol{F}\}$. The other properties are simple applications of elementary properties of conditional expectation. Q.E.D.

The following assumption is quite standard.

Assumption 2.2 : Each agent's endowment vector, denoted as $\bar{x}_i$, is semi-positive, i.e., $\bar{x}_i \in R_+{}^N$ and for some k, $\bar{x}_{ik}>0$.

The following theorem should be obvious.

Theorem 2.1 : Suppose every agent has full information. Then under the Assumptions 2.1 and 2.2, there exists an equilibrium price for each state of nature ω, except on a P-null set.

Proof: Once we apply Debreu's lemma (Debreu [4]), it is straightforward due to Proposition 2.1. Q.E.D.

Let $\tilde{e}:(\Omega,S)\rightarrow R_+^{N-1}$ represent the payoff relevant state of the environment in the sense that each agent's utility function depends on ω only through $\tilde{e}(\omega)$. For example, when agents are maximizing the wealth derived from their portfolio, $\tilde{e}(\omega)$ can be regarded as a vector of returns to their holding of risky assets in an uncertain event ω. Let E denote the set of all such states, and $E\subset R^{N-1}$ be endowed with the σ-field of events.

Assumption 2.3 : Assume $u_i\{x_i;e(\omega)\}=u_i\{\tilde{e}(\omega)x_{iF}+B_i\}$ where $x_{iF}\in R^{N-1}$ and $B_i\in R$.

Each agent is maximizing the expected utility of his wealth, or his portfolio. We notice that only markets for the linear combination of contingent claims conditional on each information-event exists. In other words, for each information-event $B(y)$, i chooses his portfolio (x_{iF}, B_i) and for eah event $\omega\varepsilon B(y)$, he gets $\tilde{e}(\omega)\cdot x_{iF}+rB_i$, though he cannot make constracts for contingent claims conditional on each event $\omega\in B(y)$ (Grossman [10]). Although this type of utility functions substantially reduce the number of contingent markets and hence reduce the opportunities for the sharing of social risk, Assumption 2.3 is based on the non-existence example presented elsewhere (Yoon [19]).

Define $\tilde{p}(\tilde{y})(\omega)=\tilde{p}(\tilde{y}(\omega))=\tilde{p}(y)$, and $\tilde{x}_i(\tilde{y})(\omega)=\tilde{x}_i(\tilde{y}(\omega))=\tilde{x}_i(y)$. Before we prove Theorem 2.2, we need the following lemma.

Lemma 2.1 : Let $\bar{x}_i=(x_{iF},B_i)$ denote the ith agent's endowment vector, so that $\bar{x}_i\in R_+^N$. Suppose $\tilde{p}(y)$ is an equilibrium price for each $y\varepsilon Y$ under Assumptions 2.1—2.3. Then

$$\sum_{i=1}^{T}\tilde{x}_{iF}(\boldsymbol{F}_i,p)=\sum_{i=1}^{T}\bar{x}_{iF}\text{ and }\sum_{i=1}^{T}\tilde{B}_i(\boldsymbol{F}_i,p)=\sum_{i=1}^{T}B_i,\quad P\text{—a.s.}$$

when $\tilde{x}_{iF}(F_i,p)$ is the optimal demand of the ith agent for the safe asset.

Proof: Let

$$A=\{\omega\in\Omega\,|\,\text{either }\sum_{i=1}^{T}\tilde{x}_{iF}(\tilde{y})(\omega)<\sum_{i=1}^{T}\bar{x}_{iF}\text{ or }\sum_{i=1}^{T}\tilde{B}_i(\tilde{y})(\omega)<\sum_{i=1}^{T}B_i\}$$

where

$$\tilde{x}_{iF}(\tilde{y})(\omega)=\tilde{x}_{iF}(\boldsymbol{F}_i,p),\quad \tilde{B}_i(\tilde{y})(\omega)=\tilde{B}_i(\boldsymbol{F}_i,p),$$

and

$\tilde{y}(\omega)=y,\ \tilde{p}(\tilde{y})(\omega)=p.$

Using the strict monotonicity of $u_i(x_i;\ \omega)$ in x_i, we can show easily that $P(A)=0$ (Yoon [19]). Q.E.D.

Assumption 2.4 : There are only two goods—a risky asset and a risk-free asset.

Define a linear ordering $\lesssim$ on Ω as follows. $\omega_1 < \omega_2$ if and only if $\tilde{e}(\omega_1) < \tilde{e}(\omega_2)$, and $\omega_1 \sim \omega_2$ if and only if $\tilde{e}(\omega_1)=\tilde{e}(\omega_2)$. Then $(\Omega, \lesssim)$ is a chain.

Assumption 2.5 : Let the distribution generated by $\tilde{e} : (\Omega, S, P) \to R_+$ be non-atomic, i.e., $P(\tilde{e}^{-1}(e))=0$ for each particular realization of e.

Assumption 2.6 : For each pair (B_k, B_l) of observable events when all the agents' information is pooled together (i.e., $B_k,\ B_l \in \boldsymbol{B}(\tilde{y}_1, \tilde{y}_2, \cdots, \tilde{y}_T)$), $\inf_{\omega \in B_l} \tilde{e}(\omega) = \underline{e}_k \leq \sup_{\omega \in B_l} \tilde{e}(\omega) = \bar{e}_l$ or $\underline{e}_k < \bar{e}_l$. Since $(\Omega, \lesssim)$ is linearly ordered, this is well defined.

The above assumption is rather restrictive and is introduced to make an equilibrium price *fully revealing* (Yoon [19]). Now we have the main theorem in this section.

Theorem 2.2 : Suppose $\boldsymbol{B}(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_T)$ is generated by $(B_m)_{m=1}^{\infty}$, $B_k \cap B_l = \phi$ for $k \neq l$ and $P(B_m)>0$, all m. Under the assumption on the utility function stated in Lemma 2.1 and under Assumptions 2.1—2.6, there exists a rational expectations equilibrium price $p(\cdot)$ such that $\boldsymbol{B}(\tilde{p}(\tilde{y}))=\boldsymbol{B}(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_T)$.

Proof: See Appendix.

III. Expectational Equilibria with Imperfect Models

1. Existence of an Expectational Equilibrium with Imperfect Models

When we defined a rational expectations equilibrium in the previous section, we assumed that every agent knows the entire function $\tilde{p}(\cdot)$ so that in a full-information equilibrium, an observation of the price is equivalent to having all information in the economy. In the equilibrium, there would be no need for recontracting after the market clearing price was announced. If all agents eventually learn the joint distribution of the equilibrium price and the underlying state of nature, they will form expectations and make consumption decisions such that this equilibrium persists over time.

To introduce a learning process of each agent, we now assume that some agents have an imperfect model of price formation process. The following definition is useful in this context.

Definition 3.1 : The model m_i of agent i is defined as a *measurable correspondence* from the price simplex S_N to the signal set Y.

Let $u_i(x_i;\ \tilde{e}(\omega))=u_i(x_i;e)$, where $\tilde{e}(\omega)=e \in E$.

Assume that each agent is maximizing $E\{u_i(x_i;e)\,|\,\tilde{y}_i=y_i\}$ subject to $px_i \leq p\bar{x}_i$. Let $\hat{x}_i(y_i, p)$ be the resulting demand function. Let $\tilde{p}(y)=p$ be such that $\sum_{i=1}^{T}\hat{x}_i(y_i, p)=\sum_{i=1}^{T}\bar{x}_i$. Let $m_i(p)=\{y_i, Y_{-i}\}\subset Y$, where $Y_{-i}=\times_{j\neq i} Y_j$ is a cartesian product of every other agent's *signal space*. Then $\tilde{p}(y)$ is still an equilibrium when each agent is maximizing $E\{u_i(x_i;e)\,|\,\tilde{y}_i=y_i,\ m_i(p)=(y_i,\ Y_{-i})\}$ for each realization y, since conditioning on $m_i(p)$ does not provide any further information. Furthermore, if we interpret $m_i(p)$ as the ith agent's model through which the information contained in the equilibrium price $\tilde{p}(y)$ is filtered, we find that the agents' models are trivially fulfilled for any realization of a joint signal y. We call this model a *naive model*.

Let an event $B_t(y)=\{\omega\epsilon\Omega\,|\,\tilde{y}_t(\omega)=y\}$, and $B_t(y_i)=\{\omega\epsilon\Omega\,|\,\tilde{y}_{it}(\omega)=y_i\}$, where $\tilde{y}_t$ (or $\tilde{y}_{it}$) denotes a joint (private) signal at time t. When the signal set Y is countable or equivalently when the sigma-field of events generated by the joint signal $\tilde{y}_t$ is countably generated for each t and the probability distribution induced by $\tilde{y}_t$ on (Ω, S, P) is independent and identical, we call $B(y)=B_t(y)$ an elementary event, if $P(B(y))>0$, all $y \in Y$. Let $m_t=(m_{1t}, m_{2t}, \ldots, m_{Tt})$ be a vector of agents' models at time t. Let m_{it}^* represent the model of agent i at time t when it is not controverted by an equilibrium and $m_t^*=(m_{it}^*)_{i=1}^{T}$. Let $\hat{p}_t(y_t;\ m_t^*)$ denote a corresponding equilibrium price at time t when the realization of the joint signal is y_t. Although $(m_{it}^*)_{i=1}^{T}$ and $\tilde{p}_t(y_t;\ m_t^*)$ are simultaneously determined, the existence of such $(m_{it}^*)_{i=1}^{T}$ is always guaranteed as we have seen when we defined a *naive model*. We now have the following definition of an expectational equilibrium with imperfect models.

Definition 3.2 : For any trading period t, and $(m_{it}^*)_{i=1}^{T}$, $\tilde{p}_t(y_t;\ m_t^*)$ is an equilibrium with a vector of fulfilled models if it satisfies the following two conditions.

(1) $\sum_{i=1}^{T}\hat{x}_i(\tilde{p}_t(y_t;\ m_t^*),\ y_{it},\ p_t^*)\leq\sum_{i=1}^{T}\bar{x}_{it}$, where $\hat{x}_i(\tilde{p}_t(y_t;\ m_t^*),\ y_{it}, p_t^*)$ maximizes $E\{u(x_i;\ e)\,|\,\tilde{y}_{it}=y_{it},\ m_{it}(p_t^*)\}$ subject to $p_t^*x_i\leq p_t^*\bar{x}_{it}$.

(2) For any $\omega \in \Omega$, $\tilde{y}_t(\omega)=y_t$ and $\tilde{p}_t(y_t;\ m_t^*)=p_t^*$, we have $\tilde{y}_t^{-1}(m_{it}(p_t^*))\cap\tilde{y}_{it}^{-1}(y_{it})\neq\phi$.

Condition (2) is stated to ensure that the private signal and the agent's model must provide mutually consistent information.

When the agents' models are defined as in Definition 3. 2, it can be shown that the effect of having a different model is reflected in the different revision of the prior distribution on the set of payoff relevant environments.

Proposition 3. 1 : Suppose every agent has a common prior, P, on (Ω, S). Then the agents' models are well-defined in the sense that the conditional probability of any Borel subset of E given y_i and $m_{ii}(p_i^*)$ is well-defined almost surely.

Proof: Define $P(W|\tilde{y}_{ii},\ m_{ii}^*(\tilde{p}_i^*))(\omega_e)=P(W|\tilde{y}_{ii}(\omega_e)=y_{ii},\ m_{ii}^*(\tilde{p}_i^*(\omega_e))=m_{ii}^*(p_i^*))$. Let $B_y=m_{ii}^*(\tilde{p}_i(\tilde{y}_i(\omega_e)))\subset Y$ and $y_i^{-1}(B_y)=B(\omega_e)$. Let $C(\omega_e)=\{\omega|\tilde{y}_{ii}(\omega)=y_{ii}(\omega_e)\}$. Then $P(W|\tilde{y}_{ii},\ m_{ii}^*(\tilde{p}_i^*))(\omega_e)=P(W|B(\omega_e)\cap C(\omega_e))$ and this is well-defined for any $W\ \varepsilon\ E, P$—a.s. Since conditional expectations as defined above do not depend on the particular versions of conditional probability we choose, our assertion holds true. Q.E.D.

In fact, we can prove a more general proposition about the existence of a prior probability measure P_i such that agent i who has an imperfect model can be viewed as if he had a perfect model with a prior distribution P_i on (Ω, S). This P_i is also uniquely determined up to P-equivalence on (Ω, S) once the model of agent i is given.

Proposition 3. 2 : Suppose the signal space Y is countable and $P(B(y))>0$, all $y\ \varepsilon\ Y$. Given $\{m_i^*\}_{i=1}^T$, there exists a unique probability measure P_i such that $E_i\{\cdot|B(y)\}=E\{\cdot|B(y_i)\cap A_i(p^*)\}$ for any $p^*=\tilde{p}^*(y;m^*)\ \varepsilon\ S_N$, where $A_i(p^*)=\tilde{y}^{-1}(m_i^*(p^*))\subset\Omega$ and $E_i\{\cdot|B(y)\}$ is the conditional expectation taken with respect to the probability measure P_i. Furthermore if P is nonatomic, P_i is also nonatomic on (Ω, S).

Proof: This is a direct consequence of the Proposition 3. 1 and Theorem 2. 2. 1 in Reyni [18]. Q.E.D.

Now we present a corollary to Theorem 2. 2.

Corollary 2. 3 : A rational expectations equilibrium with imperfect models exists under the assumptions stated in Theorem 2. 2.

Proof: Since $P\cdot\nu^{-1}$ is also nonatomic when P_i is defined from the imperfect model of agent i as in Proposition 2. 2, the rest of the proof remains the same as that of Theorem 2. 2. Q.E.D.

In fact, Corollary 2. 3 shows that $\tilde{p}^*(y_i;\ m_i^*)\neq\tilde{p}^*(y_i';\ m_i^*)$ whenever $y_i\neq y_i'$.

In other words, under the conditions stated in Corollary 2.3, agents' models always provide coarser information than the information contained in an equilibrium price given those agents' models. In technical terms, m_{it}^{*-1} is always p_t^*-measurable. We provide the following theorem to strengthen the above argument.

Let $\nu_i(\tilde{e}=e|\tilde{y}=y)=\nu_{ik}(y)=P(\tilde{e}=e|\tilde{y}_i=y_i,\ m_{it}^*(p^*)=m_{it}^*(\tilde{p}^*(y;\ m_t^*)))$.

Theorem 3.1 : Suppose E and Y are finite sets. Assume $u_i(x_i:e)=u_{i0}(x_{i0})+v_i(e\cdot x_i)$, where $e\ \varepsilon\ R_+^{N-1}$ Assume $\nu_i(e_k|y)>0$, all k, i and $y\ \varepsilon\ Y$. Then $\tilde{p}^*(y;\ m^*)\neq\tilde{p}^*(y';\ m^*)$ and such a $\tilde{p}^*(y;\ m^*)$ generically exists under the condition stated in Radner [16].

Proof: Under these assumptions, possession of a different model m_t^* is equivalent to having a different prior over the set of environments, E, as long as we are concerned about the conditional probability measure $P(\tilde{e}(\omega)=e_k|m_t^*(p^*),\ \tilde{y}_i=y_i)=\nu_i(e_k|y)$. Therefore, the proof is equivalent to Radner's proof. Q.E.D.

2. **Examples of a Learning Process**

In this section, we present two examples to clarify the definition of agents' models and to introduce our dynamic learning process. The second example is a more concrete analysis of the first example.

Example 3.1 : Suppose there are two agents—an informed agent (type I) and an uninformed (type II). Let $\Omega=\{\omega^1,\omega^2,\omega^3\}$. Assume that the signal of each agent is fixed in advance so that the type I agent (type II agent) always observes $\tilde{y}_I(\tilde{y}_{II})$ over each trading period in the future. Assume $\tilde{y}_I(\omega^1)=2$, $\tilde{y}_I(\omega^2)=3$ and $\tilde{y}_I(\omega^3)=5$ while $\tilde{y}_{II}(\omega^1)=2$, $\tilde{y}_{II}(\omega^2)=\tilde{y}_{II}(\omega^3)=4$. Then $Y_I=\{2,3,5\}$, $Y_{II}=\{2,4\}$ and $Y=Y_I\times Y_{II}$. The type I agent can distinguish every state by observing his private signal. On the other hand the type II agent cannot distinguish ω^2 and ω^3. Let ω_t denote an actual event at time t. We suppress the model of the type I agent since he is perfectly informed with his signal. The type II agent is assumed to start with a *naive* model, i.e., $m_{II1}^*(p)=\{y_{II},\ Y_I\}$ when he observes $\tilde{y}(\omega_1)=y_{II}$ and $\tilde{p}^*(\tilde{y}(\omega_1);\ m_{II}^*)=p$. Notice that this naive model is always available and not controverted by his observation of a clearing price. Suppose $\omega_1=\omega^3$ and let $\tilde{p}^*(5,4);\ m_{II1}^*)=p_1\varepsilon S_N$. At the beginning of the second period, type II agent observes $\tilde{y}_I(\omega_1)=5$ and revises his model in such a way that $m_{II2}^*(p_1)=(5,4)\varepsilon Y$ and $m_{II2}^*(p)=m_{II1}^*(p)$ for all $p\neq p_1$. Suppose $\omega_2=\omega^2$ and $\tilde{p}^*((3,4);\ m_{II2}^*)=p_2$.

Assume $p_1 \neq p_2$. At the beginning of the third period, $m_{II3}^*(p_1)=(5,4)$, $m_{II3}^*(p_2)=(3,4)$ and $m_{II3}^*(p)=m_{I1}^*(p)$, $p \notin \{p_1, p_2\}$. Let $\omega_3=\omega^3$. Then, initially $\tilde{p}^*((5,4); m_{II3}^*)=p_1$. But this conveys information to the type II agent and p_1 cannot be an equilibrium price. Therefore, the type II agent will revise his demand and a new clearing price p_3 will result. Assume also that $p_3 \neq p_1$. At $t=4$, $m_{II4}^*(p_3)=(5,4)$, $m_{II4}^*(p_2)=(3,4)$, and $m_{II4}^*(p)=m_{II1}^*(p)$, $p \notin \{p_2, p_3\}$. We notice that whenever ω^3 occurs after this period, p_3 will be an equilibrium price. This process will continue and eventually the type II agents' learning will be completed.

The implicit assumption in the above example is that each agent's consumption decision in each period is not related to the consumption decision in other periods except that his model is improving over time.

Before we get to the second example, we provide the following proposition.

Proposition 3.3 : Suppose there exists a rational expectations equilibrium under Assumptions 2.1~2.3 and the agents' utility functions are given as in Theorem 2.2. Then for any realization y, we have $E_t\{\cdot|y\}>rp^*(y; m^*)$ where $E_t(\cdot|y)$ is denoted to reflect the effect of having a different model on the revised prior on E (see Proposition 4.1).

Proof: This is due to the risk averseness of agents' utility functions (see Arrow [1]).

Remark 4.1 : If $\tilde{e}$ takes only two different values e_1 and e_2 $(e_1<e_2)$, then $e_1/r<p^*<e_2/r$ for any realizations of a joint signal.

Example 3.2 : Let $(\Omega, S)=((0,1), \boldsymbol{B})$ denotes the Borel field defined on $(0,1)$. The state contingent return $\tilde{e}(\omega)$ takes only two values, e_1 and e_2. We will assume $e_2>e_1>0$. There are two types of agents. In each period, the type I agent has a signal $\tilde{y}_{It}$ and the type II agent has a signal $\tilde{y}_{IIt}$, where $\tilde{y}_{It}: \Omega \to Y_I$ and $\tilde{y}_{IIt}: \Omega \to Y_{II}$, $Y_I=\{y_I^{(1)}, y_I^{(2)}, y_I^{(3)}\}$, $Y_{II}=\{y_{II}^{(1)}, y_{II}^{(2)}\}$ and $Y=Y_I \times Y_{II}$. Let $G=\{\omega \in \Omega | \tilde{y}_I(\omega) \neq y_I^{(3)}\}$ and assume $\tilde{y}_{II}(\omega)=y_{II}^{(1)}$, $\omega \in G$ and $\tilde{y}_{II}(\omega)=y_{II}^{(2)}$, $\omega \notin G$. $\{\tilde{y}_{It}, \tilde{y}_{IIt}\}_t$ is a stationary, independent process. Each agent has the same logarithmic utility function $\log W_t$ and they are maximizing

$$E\{\log(e \cdot x_{iF}+B_i r) | \tilde{y}_i=y_{it}, \ m_{it}^*(y_i)=p_i^*\}$$

s.t.

$$p_i^* x_{iF}+B_i \leq \bar{W}_{0i}=p^*\bar{x}_{iF}+B_i \ \text{for} \ i=I \ \text{or} \ II.$$

We assume $(\bar{x}_{iFt}, \bar{B}_{it})=(\bar{x}_{iF}, \bar{B}_i)$, all t. Then

$$\tilde{x}_{IF}(\tilde{p}_t(y_t;\ m_{IIt}^*),\ y_{It},\ p_t^*)=-\frac{\nu_I(e_I|y_t)R\bar{W}_0{}^i}{e_2-p_t^*r}+\frac{\nu_I(e_2|y_t)R\bar{W}_0{}^i}{e_1-p_t^*r}$$

where

$$\nu_I(e_1|y_t)=P(\tilde{e}=e_1|\tilde{y}_{It}=y_{It}).$$

Similarly,

$$\tilde{x}_{IIF}(\tilde{p}_t(y_t:\ m_{IIt}^*),\ y_{IIt},\ p_t^*)=-\frac{\nu_{II}(e_1|y_t)R\bar{W}_0{}^i}{e_2-p_t^*r}+\frac{\nu_I(e_2|y_t)RW_0{}^i}{e_1-p_t^*r},$$

where

$$\nu_{II}(e_I|y_t)=P(\tilde{e}=e_1|\tilde{y}_{IIt}=y_{IIt},\ m_{IIt}^*(y)=P_t^*).$$

Assume $P\{\tilde{e}=e_1|\tilde{y}_I=y_I{}^{(1)}\}>P\{\tilde{e}=e_1|\tilde{y}_I=y_I{}^{(2)}\}>P\{\tilde{e}=e_1|\tilde{y}_1=y_I{}^{(3)}\}$. Then for any t, we have $\tilde{x}_{IF}(\tilde{p}_t\ (y_t;\ m_{IIt}^*),\ y_{It}{}^{(1)},\ p_t^*)<\tilde{x}_{IF}\ (\tilde{p}_t^*(y_t:\ m_{IIt}^*),\ y_{It}{}^{(2)},\ p_t^*)<\tilde{x}_{IF}\ (\tilde{p}_t^*(y_t:\ m_{IIt}^*),\ y_{It}{}^{(3)},\ p_t^*)$, given any equilibrium price p_t^*, since $e_1/r<p_t^*<e_2/r$ for any y_t (see Proposition 3. 3).

Let $t=0$ and assume that $\tilde{y}_{I0}(\omega)=y_I{}^{(3)}$. Then $\tilde{y}_{II0}(\omega)=y_{II}{}^{(2)}$ and in this case both types of agents are informed to the same degree. Define a corresponding equilibrium price, $\tilde{p}_0^*(y_I{}^{(3)},\ y_{II}{}^{(3)}:m_{II0}^*)=p_0^*$. Also, $m_{II0}^*(p_0^*)=\{y_I{}^{(3)},\ y_{II}{}^{(2)}\}$. Suppose at $t=1$, $\tilde{y}_{I1}(\omega)=y_I{}^{(1)}$ and $\tilde{y}_{II1}(\omega)=y_{II}{}^{(1)}$. Define $m_{II1}^*(y)=p_1^*$, $p_1^*=\tilde{p}_1^*(y_1;\ m_{II}^*)$ and $m_{II1}^*(p_1^*)=\{y_I{}^{(1)},\ y_{II}{}^{(1)}\}\cup\{y_I{}^{(2)},\ y_{II}{}^{(1)}\}$. This p_1^* can be solved explicitly from the market clearing condition, i.e.,

$$\tilde{x}_{IF}(\tilde{p}_1(y_1:\ m_{II1}^*),\ y_{I1},\ p_1^*)+\tilde{x}_{IF}(\tilde{p}_1(y_1:\ m_{II1}^*),\ y_{I1},\ p_1^*)=\bar{x}_{IF}+\bar{x}_{IIF}.$$

Assume that at $t=2$, $\tilde{y}_{I1}(\omega)=y_I{}^{(1)}$ and $\tilde{y}_{II1}(\omega)=y_{II}{}^{(1)}$. The type II agent observes the realization of $\tilde{y}_{I1}$. Therefore $m_{II2}(p_1^*)=\{y_I{}^{(1)},\ y_{II}{}^{(1)}\}$. But this model is not compatible with p_1^* since as soon as $\tilde{p}_2(y_I{}^{(1)},\ y_{II}{}^{(1)}:\ m_{II2})=p_1^*$, the type II agent knows that $\tilde{y}_{I2}(\omega)=y_{I2}{}^{(1)}$ and he will decrease his demand for the risky asset while the type I agent's demand remains the same. Therefore, if p_2^* is a clearing price, than $p_2^*<p_1^*$ and $m_{II2}^*(p_2^*)=\{y_I{}^{(1)},\ y_{II}{}^{(1)}\}$. Notice that $m_{II2}^*(p_0^*)=\{y_I{}^{(3)},\ y_{II}{}^{(2)}\}$. Also for any $t>2$, if $\tilde{y}_{It}(\omega)=y_I{}^{(1)}$, then $p_t^*=p_2^*$ since now the type II agent's model is correct.

The same kind of analysis can be done for any other sequence of realizations of a joint signal $\tilde{y}_t$. We see that the price process $\{\tilde{p}_t^*\}$ is not a stationary process as long as m_{IIt}^* is not perfect. (For example, $\omega\ \varepsilon\ \Omega$ such that $\tilde{y}_1(\omega)=\tilde{y}_2(\omega)$, it was shown that $\tilde{p}_1^*(\omega)\neq\tilde{p}_2^*(\omega)$ and $\tilde{p}_1^*(\omega)$ will never be a clearing price in the future.)

Let us now briefly state what have been introduced in the above two examples. There, the type II agents' models are improved only by the observation of the successive market clearing price and the previous realization of the type I agents' signal. If that event occurs again, the price signals to type II agents that the event which occurred before is the actual event now and type II agents revise their demand correspondingly. Then there is a new equilibrium price at the event and that price is an equilibrium whenever that event occurs in the future. Agents start with a *naive model* and behave in a very honest and empiric way. We call this mode of learning a *naive learning*. The general form of a convergence of a *naive learning* process can be stated as follows:

Definition 3.3 : Let $\tilde{p}(\tilde{y}_t)$ denote a rational expectations equilibrium price when every agent has perfect model. Let $\tilde{y}_t^{-1} \cdot m_{it}^*(p^*) = A_{it}(p^*)$, and $C(p^*) = \{\omega \in \Omega \mid \tilde{p}(\tilde{y}_t)(\omega) = p^*\}$. If $\lim_{t \to \infty} P(A_{it}(p^*)\ \Delta C(p^*)) = 0$ for each $p^* \in S_N$ and for each i, then we say that each of the agent's models m_{it}^* converges simultaneously to the perfect model for each y.

The above definition has a useful economic meaning. It states that in the limit of a convergence process, the effects of *differential information* on an agent's preference formation vanishes if every agent starts with a common prior probability measure (though they initially have different private signals). The following theorem confirms the above argument.

Theorm 3.2 : Let the *naive learning* process converge to a perfect model. Assume that the σ-field of event S is countably generated. (For this definition, see Theorem 2.2.)

Let $V_i(\tilde{p}_t^*(\tilde{y}: m_t),\ \tilde{y}_i : \bar{x}_i)$ be the maximum of $E\{u_i(\tilde{x}_i(y) : e) \mid \tilde{y}_i,\ \ m_t^{*-1}(p_t^*)\}$, subject to $\tilde{p}_t^*(y : m_t^*)\ \tilde{x}_i(\tilde{p}) \leq \tilde{p}_t^*(y : m_t^*)\bar{x}_i$ for each realization of $\tilde{y} = y$. Similarly, let $V_i(\tilde{p}(y),\ \tilde{y}_i : \bar{x}_i)$ be the maximum of $E\{u_i(\tilde{x}_i(y) : e) \mid \tilde{y}_i,\ \tilde{p}(\tilde{y})\}$ s.t. $\tilde{p}(y)\ \tilde{x}_i(y) \leq \tilde{p}(y)\bar{x}_i$. Then $\lim_{t \to \infty} E\{V_i(\tilde{p}_t^*(y : m_t),\ \tilde{y}_i : \bar{x}_i)\} = E\{V_i(\tilde{p}(y),\ \tilde{y}_i : \bar{x}_i)\}$.

Proof: See Appendix.

Suppose there are only two types of agents—informed agents (type I) and uninformed agents (type II). Type I agents receive signals $\tilde{y}_{It}$ in each trading period and type II agents do not receive such a signal and observe only prices *which reflect* type I agents' information. As type II agents improve their models, type I agents become worse off and they may lose their incentives to purchase a signal $\tilde{y}_I$. From the type I agents' point of view, the existence of those agents who do not have a perfect model is necessary to collect a rent on their signal y_I. But type II agents are actually facing an informational entry barrier to this market. In other words,

$$E\{V_{II}(\tilde{p}_t(\tilde{y}),\ \bar{x}_{II})\} \geq E\{V_{II}(\tilde{p}_t^*(\tilde{y},\ m_t^*),\ \bar{x}_{II})\} \text{ for all } t.$$

The informational entry barrier, like transaction costs, constitutes an important element to limit the possible gains from the reduction of social risk through the financial markets. Existence of these markets is not free of social cost. In addition to the fixed set-up costs to open these markets, there are variable costs of expanding these markets. Transaction costs and the informational entry barrier are the major components of the variable cost of this kind. Insiders' information is often obtained at zero marginal cost and in this case, an institutional structure other than the competitive market is needed for the efficient transmission of information to those potential agents who are willing to enter these markets. The existence of those institutions can be regarded as social overhead capital since the financial market is the major means of distributing the gains from the growth of our economy.

IV. Convergence Theorem

In this section, a set of sufficient conditions for the convergence of the naive learning process is provided. We do this step by step and first show the uniqueness of an expectational equilibrium with imperfect models.

1. On the Uniqueness of an Expectational Equilibrium Price

Suppose there exists a full-information equilibrium price $\tilde{p}^*(y;\ m^*)$ for each $y \varepsilon Y$ and a corresponding vector of fulfilled models m^* which satisfy Definition 4.2. The agents' information structure is defined as $\boldsymbol{F}_i = \boldsymbol{B}(\tilde{y}_i) \wedge \boldsymbol{B}(m_i^*(\tilde{p}))$. Let us decompose an economy $\boldsymbol{E}_F = (u_i,\ \bar{x}_i,\ \boldsymbol{F}_i)_{i=1}^T$ into a family of elementary economies $\{\boldsymbol{E}_k\}_{k=1}^{\infty}$, $\boldsymbol{E}_k = (u_i,\ \bar{x}_i,\ B(y^k))$ where $\boldsymbol{E}_F$ denotes a full-information economy and $B(y^k)$ is an elementary event introduced before and $Y = \{y^k\}_{k=1}^{\infty}$ with $P(B(y^k)) > 0$ for all k. In each elementary economy $\boldsymbol{E}_k$, each agent is

$$\text{maximizing} \quad E_i\{u_i(\tilde{e}(\omega)x_i + B_i r) \mid B(y^k)\}$$

subject to　　$p_k{}^* x_i + B_i = p_k{}^* \bar{x}_i + \bar{B}_i$,

where $E_i(\cdot | B(y^k)) = E(\cdot | B(y_i{}^k) \cap A_i(p_k{}^*))$. (For the definition of $A_i(p_k{}^*)$, see Definition 4.3.) If $p_k{}^*$ is an equilibrium price for this economy, then $p_k{}^* = \tilde{p}^*(\tilde{y}(\omega_k); m^*)$ for any $\omega_k \in B(y^k)$.

Let us fix $E_i(\cdot | B(y^k))$ and vary the price in each elementary economy as follows. Each agent is now maximizing $E_i\{u_i(\tilde{e}(\omega)x_i + B_{i}r) | B(y^k)\}$ subject to $p_k x_i + B_i = p_k \bar{x}_i + \bar{B}_i$. If the equilibrium price for this economy is unique, then $p_k{}^*$ must be the only equilibrium price which gives each agent the additional information $A_i(p_k{}^*)$. We defined an elementary economy in such a way that information is assumed to be given and they do not utilize informaton contained in the price. The elementary economy is an artificial economy; but the equilibrium in the elementary economy, $\boldsymbol{E}_k$ is an equilibrium of a full-information economy at each elementary event, $B(y^k)$. Suppose there are two different equilibria in the full-information economy. Then in each elementary economy, there will also be two equilibria. In other words, if the equilibrium in each elementary economy is unique, then the full-information equilibrium is also unique, for given m^*.

Let $x_i(k, p) = (x_{iF}(k, p),\ B_i(k, p))$ denote the demand function in each elementary economy, $\boldsymbol{E}_k$. By showing that the demand function in the elementary economy exhibits the gross substitute property, we prove that the equilibrium in each artificial economy is unique. We say that $x_i(k, p)$ has a gross substitute property if

$$\frac{dx_{iF}(k, p)}{dp} < 0 \text{ and } \frac{dB_i(k, p)}{dp} > 0,$$

since the price of the safe asset is normalized to be one, and the information structure in each elementary economy is given and does not depend on p.

Following Arrow [1], define

$$R_A(W_i) = -\frac{U_i''(W_i)}{U_i'(W_i)}$$

as the absolute risk aversion and $W_i \cdot R_A(W_i) = R_R(W_i)$ as the relative risk aversion when agent i holds his wealth W_i.

Assumption 4.1: $R_A(W_i)$ is a decreasing function of W_i, and $R_R(W_i)$ is an increasing function of W_i.

Using this assumption, we have the following lemma presented in Arrow [1]. Though we present this lemma in a different context, the connection should be obvious.

Lemma 4.4: Under the above assumption both risky asset and safe asset are normal goods in each elementary economy. In other words, as W_i increases, the demand for both types of asset increases.

Notice that the wealth W_i of agent i depends on the clearing price p and if his endowment $(\bar{x}_{iF}, \bar{B}_i)$ is given, we are not sure how changes in the demand for both types of assets are related to the changes in the wealth induced by changes in p. The following lemmas are quite useful in this context.

Lemma 4.5 (Fisher [6]): If all goods are normal, a necessary and sufficient condition for the endowment-constant demand curves to exhibit gross substitutes is that the wealth-constant demand curves do so.

Using this lemma under Assumptions 2.3 and 2.4, we prove gross substitutability when the initial wealth in monetary terms is fixed. Then the desired result that the demand for both types of assets exhibits gross substitutes can be obtained.

Lemma 4.6: When the wealth W_i of agent i is exogenously given, the risky asset and the safe asset are gross substitutes in each elementary economy, $\boldsymbol{E}_k$.

Proof: Each agent is maximizing $E_i\{u_i(\bar{w}_i+x_i(\tilde{e}(\omega)-rp))\,|\,B(y^k)\}$. Suppose p changes to p' and define $\tilde{R}(\omega)=\tilde{e}(\omega)-rp$. As p changes, $\tilde{R}(\omega)$ changes. But this change in the rate of return is a linear shift and the demand for the risky asset declines if $p'>p$ when the risky asset is a normal good (Arrow [1]). Q.E.D.

Now we have the following theorem.

Theorem 4.1: Under the Assumptions 2.1—2.6 and Assumption 4.1, a rational expectations equilibrium with imperfect models exists and is unique.

Proof: Existence is proved in Corollary 2.2. Uniqueness follows from Lemmas 4.4, 4.5 and 4.6. Q.E.D.

2. Convergence Theorem

To prove that the models of the agents converge to a perfect model, we need a series of assumptions. The following two assumptions are maintained in this paper.

Assumption 4.2: The joint distribution generated by $\{\tilde{e}_t, \tilde{y}_t\}$ is independent and identically distributed over time. Also, $\boldsymbol{B}(\tilde{y}_t)$ is generated by a countable number of elementary events.

Assumption 4.3: Every agent knows the information structure of every other agent. In other words, the distribution induced by $\tilde{y}_t$ is known to all agents. Further, they observe the realization of $\{\tilde{y}_{it}, \tilde{p}_t^*, \tilde{y}_{t-1}\}$ in each period t.

The acquisition and use of information certainly requires the use of resources and the choice of information structure needs to be incorporated into our formal framework. The signal that agent i purchases as well as the market clearing price in each period constitutes his information structure if he possesses a perfect model. This signal $\tilde{y}_i$ is interpreted as the private information source of agent i and will be fixed throughout the future trading periods. Then buying a certain signal at the beginning of the trading process can be regarded as an investment activity and the optimal choice of a signal needs complete knowledge of the price formation process, including a knowledge of the signals and models possessed by other agents. Although this requires extraordinary rationality on the part of each agent, the assumption that the agents know each other's information structures is consistent with our reasoning. Since the realization of a joint signal is not known contemporaneously, agents will have incentives to improve their models to process the information conveyed by the price.

The assumption that each agent can observe the previous realization of the joint signal seems to be rather restrictive. But this can be viewed as a good approximation to the real situation in the security market. The function $\tilde{e}_t(\cdot)$ (or $\tilde{e}(\cdot)$ since it is identically distributed over time) is known to every agent though its realization is not observable in the current period. If it is interpreted as a vector of returns to the holding of risky assets, $\tilde{e}_{t-1}(\omega)$ will be observable. If it generates a strictly finer partitioning on Ω than $\tilde{y}_{t-1}$, $\tilde{y}_{t-1}$ can certainly be inferred from $\tilde{e}_{t-1}(\omega)$. If the underlying state of nature ω itself is directly observable after one period, it gives more information than the observation of $\tilde{y}_{t-1}(\omega)$. A signal may be information about the stock split decision of a particular class of firms in the next period. It may also be information about the merger movements in near future or about the governmental regulation on the nominal rate of interest on savings deposits, which constitutes a safe asset in the absence of inflation. Agents also obtain information about the changes in the stock of money and form the expectations of the future price level to adjust their portfolio. Assumption 4.2 is tantamount to the assumption that this information in the economy is revealed to every agent after the lapse of time. Our emphasis on this assumption is motivated by the fact that the equilibrium price depends on the realization of the joint signal and not directly on the underlying state since the latter is revealed after the exchange is completed. What agents can do best is to understand how the equilibrium price is related to the joint signal so that they can use the foreknowledge about the underlying state which is indicated by the price. To learn this relationship they need a series of past observations about the price and its information content and once their learning is completed, the self-fulfilling equilibrium will persist over time.

The other assumption that we have made implicit in the examples of a naive learning process is that economic agents do not have any motive for saving. This assumption is rather restrictive since we exclude an intertemporal optimizing behavior. Although this aspect of the learning process was studied elsewhere (Yoon [19]), we make this restrictive assumption explicit in this paper.

Assumption 4.4: Each agent has a myopic horizon and consumes everything that he has at the end of the period.

The following four conditions have been proved so far.

(1) $\tilde{p}_t^*(y;\ m_t^*) \neq \tilde{p}_t^*(y';\ m_t^*)$ if $y \neq y'$,

(2) $\tilde{p}_t^*(y;\ m_t^*) \neq \tilde{p}_t(y')$, if $y \neq y'$, where $\tilde{p}_t$ denotes a strict rational expectations equilibrium price, and m_t^* is imperfect,

(3) $\tilde{p}_t(y) \neq \tilde{p}_t(y')$ if $y \neq y'$,

(4) both $\tilde{p}_t^*(y;\ m_t^*)$ and $\tilde{p}_t(y)$ is unique for each realization y.

Conditions (1) and (3) are satisfied by Corollary 2.3 and Theorem 3.1, Condition (4) is also satisfied by Theorem 4.2 and Theorem 3.1. Condition (2) needs an explanation. For two different events $B(y)$ and $B(y')$, define a new model m_t^{**} as follows; $m_t^{**}(p) = m_t^*(p)$ in the event $B(y)$ and

$m_t^{**}(\tilde{p}(y'))=y'$ in the event $B(y')$. If we imagine that a representative agent i has a model m_{it}^{**} rather than m_{it}^{*}, both Corollary 2.3 and Proposition 4.2 satisfy condition (4) in the way they satisfy condition (1).

The following lemma is provided to prove that agents have enough (infinite) opportunities to revise their models over time.

Lemma 4.7: Let B_{lt} be an elementary event B_l when it occurs at t. Each elementary event B_{lt} occurs infinitely often. In other words, $P(B_{lt},\ \text{i.o.})=1$, where $P(B_{lt},\ \text{i.o.})=P(\limsup_t B_{lt})=P(\bigcap_{t'=1}^{\infty}\bigcup_{t=t'}^{\infty} B_{lt})=1$ and $B_l=\{\omega \in \Omega \mid \tilde{y}_t(\omega)=\tilde{y}_t(\omega_l),\ \tilde{y}_t(\omega_l)=y\}$.

Proof: Since $\{\tilde{y}_t\}$ is a stationary independent process, events $B_{lt}=\{\omega \in \Omega \mid \tilde{y}_t(\omega)\}$ and $B_{lt}'=\{\omega \in \Omega \mid \tilde{y}_t(\omega)=\tilde{y}_t(\omega_l)\}$ are independent for all t and l. Since $P(B_{lt})=P(B_{lt}')=P(B_l)>0$, $\sum_{t=1}^{\infty} p(B_{lt})=\infty$ for each l. Using the Borel-Cantelli lemma (Chung [3], pp. 71-79), we see that $P(B_{lt},\ \text{i.o.})=1$. That is, each elementary event B_l occurs infinitely often with probability 1. Q.E.D.

Before we get to our main theorem, we emphasize again that the agents are assumed to be empiricists and very honest about their model and once they observe the new informaton conveyed by the price through their models, recontracting is allowed until their models are fulfilled in each period.

Now we have the following convergence theorem.

Theorem 4.2: If economic agents adopt a *naive learning*, a sequence of expectational equilibria with imperfect models converges to a full-information equilibrium as defined in Section II, under the assumptions maintained either in Theorem 4.1 or in Theorem 3.1, and under Assumptions 4.2 ~4.4.

Proof: Consider an elementary event $B_k=B(y^k)$ as defined in Definition 4.2. Suppose t_k is the first time that event B_k occurs. Let $\tilde{p}_{t_k}^{*}(y^k;\ m_{t_k})=p_{t_k}$ and $m_{it_k}^{*}(p_{t_k})=\{y_{it}^k,\ Y_{-i}\}$ for all i since agents do not know the relationship between the equilibrium price at the elementary event B_k and the realization of the joint signal y^k at time t_k. Notice that if $m_{it}^{*}(p_{t_k}^{*})=y_t^k=y^k$ for all i, then for any $\omega \in B^k$, $\tilde{p}_t^{X}(\omega)=\tilde{p}_t(y^k)$, where $\tilde{p}_t(y^k)$ is a strict rational expectations equilibrium price, since every agent is maximizing his utility conditioning on the elementary event B^k which is provided by his model. At $t=t_k+1$, agent i can revise his model in such a way that $m_{i,t_k+1}(p_{t_k})=y^k$ and this will be true for any agent (agents revise their models only after the observation of the realization of the previous joint signal and the corresponding price). Since an elementary event B_{tk} occurs infinitely often by Lemma 4.7, there exists t_k' such that $\tilde{y}_{t_k}'(\omega)=\tilde{y}_{t_k}(\omega)$ for $\omega \in B^k$. But then at $t=t_k'$, p_{t_k} cannot be an equilibrium price, since now every agent knows that p_{t_k} contains information, $\tilde{y}_{t_k}'(\omega)=\tilde{y}_{t_k}(\omega)$. This is because Conditions (1)$\sim$(4) are satisfied under the assumptions stated in the above theorem. Let $p_{t_k}'^*$ be a new equilibrium price when all agents know that event B^k occurs. Then $p_{t_k}'^*=p_{t_k}(y^k)$ and for any $t>t_k'$, $m_{it}^*(p_{t_k}')=y^k$ for all i and $\tilde{p}_t^*(\omega)=\tilde{p}_t(\omega)$ for all $\omega \in B^k$. Since this is ture for all elementary events, our assertion is true. Q.E.D.

V. Concluding Comments

We have introduced an expectational equilibrium with imperfect models and examined the existence and uniqueness of this equilibrium. A *naive learning* process and the convergence theorem for the dynamic adjustment of expectational equilibria have been presented. The idea that each agent has his own model of price formation was also discussed by Radner [17], though we take a different framework.

What we have also emphasized is that fixed costs are an important element in the acquisition of information and the cost of purchasing information must be regarded as an investment in the instruments which provide useful information over time (i.e., the signal $\tilde{y}_t$). The rationality requirement seems to be closely related with this view on the cost of information. The development of financial intermediaries such as a Mutual Fund or an Investment Bank can be regarded as economizing on rationality requirements (Yoon [20]).

Grossman [10] discussed the cost of information in the context of a dimensionality of the message space. His point is that even in a self-fulfilling long-run equilibrium the price system may not fully reflect insiders' information if there are some non-traded assets (e.g., human capital or small businesses) due to institutional constraints. In this paper, we expressed the view that the incentive for the informed agents to acquire their costly signal is derived from the opportunity to collect a rent to their information endowment through the dynamic adjustment process.

Appendix

(1) *Proof of Theorem* 2.2: We will first assume that each agent has full information, $\boldsymbol{B}(\tilde{y}_1, \ldots, \tilde{y}_T)$. Then we will show that a full information equilibrium price $\tilde{p}_I(\tilde{y})$ generates a sigma-field of events finer than $\boldsymbol{B}(\tilde{y}_1, \ldots, \tilde{y}_T)$. Since any equilibrium price cannot contain more information than $\boldsymbol{B}(\tilde{y}_1, \ldots, \tilde{y}_T)$, we can conclude that $\boldsymbol{B}(\tilde{p}_I(\tilde{y}))=\boldsymbol{B}(\tilde{y})$. Then $\boldsymbol{B}(\tilde{y}_i, \tilde{p}_I(\tilde{y}))=\boldsymbol{B}(\tilde{y})$ and so there exists a rational expectations equilibrium price $\tilde{p}(y)=\tilde{p}_I(y)$. Let $B_k, B_l \in \boldsymbol{B}(\tilde{y})$, $B_k \cap B_l=\phi$. Suppose for $\omega_k \in B_k$ and $\omega_l \in B_l$, $\tilde{p}_I(\tilde{y})(\omega_k)=\tilde{p}_I(\tilde{y})(\omega_l)=p$. By Assumptions 2.5 and 2.6 we have

$$E\{eU_i'(e\cdot\tilde{x}_{iF}(y)+B_i(y)R)\,|\,B_l\}\leq E\{\bar{e}_l U_i'(e\cdot\tilde{x}_{iF}(y)+B_i(y)R)\,|\,B_l\}$$
$$=\bar{e}_l E\{U_i'(e\cdot\tilde{x}_{iF}(y)+B_i(y)R)\,|\,B_l\},$$

since $P\cdot e^{-1}$ is non-atomic and $P(B_l)>0$. Similarly,

$$E\{eU_i'(e\cdot\tilde{x}_{iF}(y)+\tilde{B}_i(y)R\,|\,B_k\}>\underline{e}_k E\{U_i'(e\cdot\tilde{x}_{iF}(y)+\tilde{B}_i(y)R\,|\,B_k\}.$$

Suppose

$$\tilde{x}_i(\tilde{y})(\omega_k)=(\tilde{x}_{iF}(y),\ B_i(y))=x_i(\boldsymbol{F}_i,p) \text{ at } \omega=\omega_k.$$

Let $p=(P_F,\ P_B)$. Then

$$\frac{P_F}{P_B}=\frac{E\{eU_i'(e\cdot\tilde{x}_{iF}(y)+B_i(y)R)\,|\,B_k\}}{RE\{U_i'(e\cdot\tilde{x}_{iF}(y)+B_i(y)R)\,|\,B_k\}}>\frac{\underline{e}_k}{R}.$$

But

$$\frac{\underline{e}_k}{R}\geq\frac{\bar{e}_l}{R}>\frac{E\{eU_i'(e\cdot\tilde{x}_{iF}(y)+B_i(y)R)\,|\,B_l\}}{RE\{U_i'(e\cdot\tilde{x}_{iF}(y)+B_i(y)R)\,|\,B_l\}}.$$

Therefore, at $\omega=\omega_l \in B_l$, the ith agent's demand for the risky asset must decrease if the same price prevails. This is true for for all i. Then

$$\sum_{i=1}^{T}\tilde{x}_{iF}(y)(\omega_k)>\sum_{i=1}^{T}\tilde{x}_{iF}(y)(\omega_l) \text{ at } P_I(y)(\omega_k)=P_I(y)(\omega_l)=p.$$

Similarly,

$$\sum_{i=1}^{T}B_i(y)(\omega_k)<\sum_{i=1}^{T}B_i(y)(\omega_k).$$

By Lemma 2.1, we know that

$$\sum_{i=1}^{T}\tilde{x}_{iF}(y)(\omega_k)=\sum_{i=1}^{T}\bar{x}_{iF},$$

and

$$\sum_{i=1}^{T}B_i(y)(\omega_k)=\sum_{i=1}^{T}\bar{B}_i.$$

Therefore p cannot be an equilibrium price at $\omega=\omega_l \in B_l$. Now we have proved that $\boldsymbol{B}(p_I(y))=\boldsymbol{B}(y)$. By Theorem 2.1, we can construct an equilibrium price p_I such that $\boldsymbol{B}(p_I)=\boldsymbol{B}(y)$. Q.E.D.

(2) *Proof of Theorem* 3.2: Before we provide a proof, we need following two well known lemmas. We state these without proof.

*Lemma A.*1: Let the sequence of economies $e_t=(u_{it}, \bar{x}_{it})_{i=1}{}^T$ converge to $e_0=(u_{i0}, \bar{x}_{i0})_{i=1}{}^T$, where u_{it} converges to u_{i0} in the topology of uniform convergence and each u_{it} satisfies Assumption 3.2. Let the corresponding sequence of the equilibrium price correspondence be $A_t(e)$. Then the correspondence A from the space of economies to the price simplex is upper-semicontinuous.

*Lemma A.*2: Let $v_n(p_n, \bar{x})$ be the maximum of $u_n(x)$ subject to $p_k x \leq p_k \bar{x}$. If u_n converges to u_0 and when p_k converges to p_0, then $v_n(p_k, \bar{x})$ converges to $v_0(p_k, \bar{x})$ if $\{u_n(x)\}$ is a sequence of a strictly concave and continuous functions.

Let

$$D_t{}^i(\omega_e)=\{\omega \in \Omega \,|\, y_{it}(\omega)=y_{it}(\omega_e),\ m_t{}^*(p_t{}^*(y:m_t{}^*)(\omega_e))=m_t{}^*(p_t{}^*(y:m_t{}^*)(\omega)\}.$$

$(D_t{}^i(\omega_e)\neq\phi$ by the definition of an equilibrium introduced in Definition 3.2.) Then

$$E\{u_i(x_i:e)\,|\,y_i, m_{it}(p_t{}^*)\}(\omega_e)=\frac{1}{p(D_t{}^i(\omega_e))}\int_{D_t{}^i(\omega_e)} u_i(\bar{x}(y):e)\ P(d\omega).$$

Since $\{m_{it}\}_t$ converges to p^* for all i, $P(D_t{}^i(\omega_e))\rightarrow P(D(\omega_e))$, where $D(\omega_e)=\{\omega \in \Omega\,|\,y(\omega)=y(\omega_e)\}$. If we introduce a pseudometric ρ defined on (Ω, S, P) by $\rho(A,B)=P(A\triangle B)$ for $A, B\in S$, then the integral is a continuous function on this pseudometric space $M(S,P)$ (See [3], p. 44).

Therefore

$$\int_{D_t{}^i(\omega_e)} u_i(x_i:e)\ P(d\omega) \quad\rightarrow\quad \int_{D(\omega_e)} u_i(x_i:e)\ P(d\omega) \quad \text{as } t\rightarrow\infty$$

and $v_i(p_t{}^*(y:m_t{}^*), y_i:\bar{x}_i)$ converges almost surely to $v_i(p(y), y_i:\bar{x}_i)$ due to Lemmas A.1 and A.2. Since $u_i(x_i:e)$ was assumed to be a bounded function, $\lim_{t\rightarrow\infty} E\{v_{it}(p_t{}^*(y:m_t{}^*), y_i:\bar{x}_i)\}=E\{v_i(p^*(y), y_i:\bar{x}_i)\}$ (See [3], p. 67). Q.E.D.

References

[1] Arrow, K.J., *Essays on the Theory of Risk-Bearing*, Chicago: Markham, Amsterdam: North-Holland, 1971.

[2] Brock, H., "On Models of Expectations that Arise from Maximizing Behavior of Economic Agents Overtime," *Journal of Economic Theory*, 5.

[3] Chung, K.L., *A Course in Probability Theory*, Academic Press, Inc., New York, 1974.

[4] Debreu, G., *Theory of Value*, Yale University Press, New Haven, 1959.

[5] Fama, E.F., "*Efficient Capital Market:* A Review of Theory and Empirical Works," *Journal of Finance*, 25, pp. 383-417.

[6] Fisher, F.M., "Gross Substitute and the Utility Function," *Journal of Economic*

Theory, 4, pp. 82-87.

〔7〕 Green, J.R., "Information, Efficiency and Equilibrium," Discussion Paper No. 284, Harvard Institute of Economic Research, Harvard University, April 1971.

〔8〕 Green, J.R., "The Non-existence of Informational Equilibria," Discussion Paper No. 410, Harvard Institute of Economic Research, Harvard University, April 1975.

〔9〕 Grossman, S., "Rational Expectations and the Econometric Modelling of Markets Subject to Uncertainty," *Journal of Econometrics*, 3, pp. 255-272.

〔10〕 Grossman, S., "Further Results on the Informational Efficiency of Competitive Stock Markets," Research Paper No. 377, Graduate School of Business, Stanford University, 1977.

〔11〕 Hahn, F., *On the Notion of Equilibrium in Economics*, Cambridge University Press, 1973.

〔12〕 Jordan, J.S., "Informational Temporary Equilibria," Discussion Paper No. 77-89, University of Minnesota, April 1977.

〔13〕 Kihlstrom, R.E., and L.J. Mirman, "Information and Market Equilibrium," *Bell Journal of Economics*, 6, pp. 357-376.

〔14〕 Kreps, D.M., "A Note on Fulfilled Expectations Equilibria," *Journal of Economic Theory*, 14.

〔15〕 Lucas, R.E. Jr., and E.C. Prescott, "Investment under Uncertainty," *Journal of Economic Theory*, 4, pp. 659-681.

〔16〕 Radner, R., "Rational Expectations Equilibrium; Generic Existence and the Information Revealed by Prices," Working Paper No. 19-245, Center for Research in Management Science, University of California, Berkeley, revised, 1977.

〔17〕 Radner, R., "Rational Expectations Equilibrium with Imperfect Models," mimeo., University of California, Berkeley, 1977.

〔18〕 Radner, R., and J.S. Jordan, "Non-existence of a Rational Expectations Equilibrium," mimeo., IMSSS, Stanford University, 1977.

〔19〕 Renyi, A., *Foundations of Probability*, Holden-Day, Inc., San Francisco, 1970.

〔20〕 Yoon, C.H., *Rational Expectations Equilibrium, Cost of Information and Welfare*, unpublished Ph. D. dissertation, Stanford University, 1978.

〔21〕 William, B., and F.T. Dolbear, "Social Risk and Financial Markets," *American Economic Review*, 1968.